

Homework Set 7: Derivatives of Algebraic Functions

1. Use the definition of the derivative to find $\frac{d}{dx}\left[\frac{1}{x}\right]$
 2. Use the definition of the derivative to find $\frac{d}{dx}\left[\frac{1}{\sqrt{x}}\right]$
- } see attached

Find $\frac{dy}{dx}$:

3. $y = 4x^7$ $\frac{dy}{dx} = 4 \cdot \frac{d}{dx}[x^7] = 4 \cdot 7x^6 = 28x^6$

4. $y = -\frac{1}{3}(x^3 - 2x + 4)$ $\frac{dy}{dx} = -\frac{1}{3} \frac{d}{dx}[x^3 - 2x + 4] = -\frac{1}{3}(3x^2 - 2) = -x^2 + \frac{2}{3}$

5. $y = \pi x - \frac{2}{\pi}$ $\frac{dy}{dx} = \frac{d}{dx}[\pi x] - \frac{d}{dx}\left[\frac{2}{\pi}\right] = \pi - 0 = \pi$

6. $y = -3x^5 + x^4 - 2x + 7$ $\frac{dy}{dx} = -3 \cdot 5x^4 + 4x^3 - 2 + 0 = -15x^4 + 4x^3 - 2$

Find $f'(x)$:

7. $f(x) = \sqrt{x} + \frac{1}{x}$ $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$ ← from problem 1.3

8. $f(x) = x^{-3} + \frac{1}{x^4}$ $f'(x) = -3x^{-3-1} + -4x^{-4-1} = -3x^{-4} - 4x^{-5} = -\frac{3}{x^4} - \frac{4}{x^5}$

9. $f(x) = 5x^8 + 4x^{-5} - 8\sqrt{x} - \sqrt{2}$ $f'(x) = 5 \cdot 8x^7 + 4 \cdot -5x^{-6} - 8 \cdot \frac{1}{2\sqrt{x}} - 0$
 $= 40x^7 - 20x^{-6} - 4/\sqrt{x}$

10. Find $\frac{dx}{dt}$ if $x = 5t^3 + 2t$ $\frac{dx}{dt} = 5 \cdot 3t^2 + 2 \cdot 1 = 15t^2 + 2$

11. Find $\Theta'(\varepsilon)$ if $\Theta = 4\pi\varepsilon^2$ $\Theta'(\varepsilon) = 4\pi \cdot 2\varepsilon^1 = 8\pi\varepsilon$

12. Find $f'''(2)$ if $f(x) = 1 - x^7$ $f'(x) = 0 - 7x^6 = -7x^6$ $f''(x) = -42x^5$ $f'''(x) = -210x^4$

13. Find $\frac{d^2y}{dx^2}$ if $y = 4(x^3 - x^2)$ $\frac{dy}{dx} = 4(3x^2 - 2x)$
 $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = 4(3 \cdot 2x - 2) = 4(6x - 2)$

Find $f'(x)$:

$$f'(x) = (3x - x^5)'(4x - 1) + (2x^2 - x + 5)(3 - 5x^4)'$$

$$14. f(x) = (3x - x^5)(2x^2 - x + 5)$$

$$15. f(x) = (4x^7 + 2x^2 - 1)(3x^2 - x + 5) \quad f'(x) = (4x^7 + 2x^2 - 1)'(3x^2 - x + 5) + (3x^2 - x + 5)'(4x^7 + 2x^2 - 1)$$

$$16. f(x) = \left(\frac{1}{x} + x^{-2}\right)(2x^2 - 3x + 5) \quad f'(x) = \left(\frac{1}{x} + x^{-2}\right)'(2x^2 - 3x + 5) + (2x^2 - 3x + 5)' \left(\frac{1}{x} + x^{-2}\right)$$

$$17. f(x) = \frac{1 - x - x^2}{2x + 1} \quad f'(x) = \frac{(2x + 1)'(1 - x - x^2) - (1 - x - x^2)'(2x + 1)}{(2x + 1)^2}$$

Find $\frac{dy}{dx}$:

$$18. y = \frac{x^5 - 3x^2}{x^2 + 1} \quad \frac{dy}{dx} = \frac{(x^2 + 1)'(x^5 - 3x^2) - (x^5 - 3x^2)'(x^2 + 1)}{(x^2 + 1)^2}$$

$$19. y = \frac{(2\sqrt{x} + x)(x^3 - 2x)}{x + 4} \quad \frac{dy}{dx} = \frac{(x + 4)[(2\sqrt{x} + x)'(x^3 - 2x) + (x^3 - 2x)'(2\sqrt{x} + x)] - (2\sqrt{x} + x)(x^3 - 2x)'(x + 4)}{(x + 4)^2}$$

$$20. y = (3x^{-2} - 4x^7) \left(\frac{x^2 + 1}{x - 1} \right) \quad \frac{dy}{dx} = (3x^{-2} - 4x^7)' \left(\frac{x^2 + 1}{x - 1} \right) + (3x^{-2} - 4x^7) \left(\frac{x^2 + 1}{x - 1} \right)'$$

$$21. y = (7 - x^{-3}) \left((x^5 - \sqrt{x}) \left(x^2 - \frac{1}{x^2} \right) \right) \quad \frac{dy}{dx} = (7 - x^{-3})' \left((x^5 - \sqrt{x}) \left(x^2 - \frac{1}{x^2} \right) \right) + (7 - x^{-3}) \left((x^5 - \sqrt{x})' \left(x^2 - \frac{1}{x^2} \right) + (x^5 - \sqrt{x}) \left(x^2 - \frac{1}{x^2} \right)' \right)$$

$$22. \text{ Find all values of } x \text{ for which the tangent line to } y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 1 \text{ is horizontal.}$$

$$23. \text{ Find all values of } x \text{ for which the tangent line to } y = \frac{x^2 + 9}{x} \text{ is horizontal.}$$

$$24. \text{ Show that the segment of the tangent line to the graph of } y = \frac{1}{x}, \text{ that is cut off by the coordinate axes is bisected by the point of tangency.}$$

$$25. \text{ What should it mean to say that two curves intersect at right angles?}$$

$$26. \text{ Show that the curves } y = \frac{1}{x} \text{ and } y = \frac{1}{2 - x} \text{ intersect at right angles.}$$

see
attached.

Homework Set 7 Solutions

$$1) \frac{d}{dx} \left[\frac{1}{x} \right] = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{(x+h)x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} \cdot \frac{1}{\cancel{x}} \cdot \frac{1}{\cancel{x}} \\ = \lim_{h \rightarrow 0} \frac{-h}{x(x+h) \cdot h} = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)^0} = -\frac{1}{x^2}$$

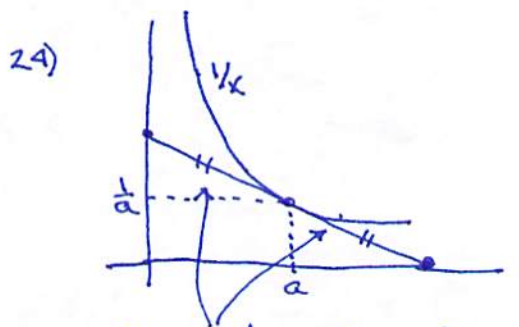
$$2) \frac{d}{dx} \left[\frac{1}{\sqrt{x}} \right] = \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \\ = \lim_{h \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \lim_{h \rightarrow 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\ = \lim_{h \rightarrow 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{-1}{x(2\sqrt{x})} = \frac{-1}{2x^{3/2}}$$

22) horizontal line $\Rightarrow$ slope $= 0 \quad \therefore m_{\text{tan}} = \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{2} \cdot 3x^2 - \frac{1}{2} 2x - 6 + 0 = x^2 - x - 6 \stackrel{!}{=} 0 \Rightarrow (x-3)(x+2) = 0 \\ x-3=0 \quad \text{or} \quad x+2=0 \\ x=3 \quad \quad x=-2$$

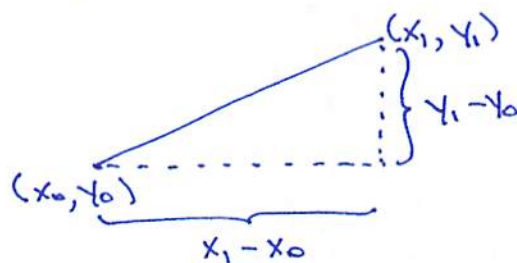
$\therefore$ the tangent line is horizontal at $x=3$ and $x=-2$

23) As above: $\frac{dy}{dx} = \frac{x(2x) - (x^2+9)(1)}{x^2} \stackrel{!}{=} 0 \Rightarrow 2x^2 - x^2 - 9 = 0 \\ x^2 - 9 = 0 \\ x = \pm 3$



Show these two line segments have the same length.

Sol'n The formula for length of a line segment is $\sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}$



So to find the lengths of the segments we will need the coordinates of their end points, i.e. the axis intercepts

cont. $\rightarrow$

The slope of the tangent line, $m_{\tan}$, is: $m_{\tan} = \frac{-1}{x^2} = \frac{-1}{a^2}$

Since it passes through the point $(a, 1/a)$ its equation is:

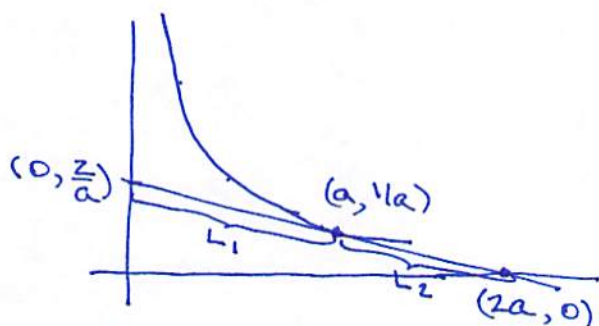
$$\frac{y - 1/a}{x - a} = \frac{-1}{a^2} \Rightarrow y - \frac{1}{a} = -\frac{x-a}{a^2} \Rightarrow y = -\frac{x}{a^2} + \frac{1}{a^2} + \frac{1}{a} = -\frac{x}{a^2} + \frac{2}{a}$$

So the equation of the tangent line is $y = -\frac{x}{a^2} + \frac{2}{a}$

Its y-intercept is $\frac{2}{a}$ and its x-int. is: $0 = -\frac{x}{a^2} + \frac{2}{a}$

$$\frac{x}{a^2} = \frac{2}{a} \Rightarrow x = \frac{2a^2}{a} = 2a$$

So we have:



$$\begin{aligned} L_1 &= \sqrt{(1/a - 2/a)^2 + (a - 0)^2} \\ &= \sqrt{(-1/a)^2 + a^2} \\ &= \sqrt{1/a^2 + a^2} \end{aligned}$$

$$\begin{aligned} L_2 &= \sqrt{(0 - 1/a)^2 + (2a - a)^2} \\ &= \sqrt{1/a^2 + a^2} \end{aligned}$$

Since $L_1 = L_2$ we have proven that the line segments have the same length.

- 25) If two curves intersect at right angles it should mean their tangent lines are perpendicular:



26) $y = \frac{1}{x} \Rightarrow m_{\tan} = \frac{dy}{dx} = -\frac{1}{x^2}$

$y = \frac{1}{2-x} \Rightarrow m_{\tan} = \frac{dy}{dx} = \frac{(2-x)(0) - (1)(-1)}{(2-x)^2} = \frac{(2-x)(0) - (1)(-1)}{(2-x)^2} = \frac{1}{(2-x)^2}$

The point of intersection occurs when $\frac{1}{x} = \frac{1}{2-x} \Rightarrow 2-x = x \Rightarrow 2 = 2x \Rightarrow x=1$

And at $x=1$ $m_{\tan}(1/x) = -\frac{1}{1^2} = -1$ and $m_{\tan}(1/(2-x)) = 1/(2-1)^2 = 1$

but since these slopes are negative reciprocals it means the lines are perpendicular.