

Homework set 13 Solutions

1) Solve $x^3 - 6 = 0$ numerically

Solution $f(x) = x^3 - 6 \Rightarrow f'(x) = 3x^2 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 - 6}{3x_n^2}$

1st guess: $x_1 = 2$

$$x_2 = 2 - \frac{2^3 - 6}{3 \cdot 2^2} = 1.8\bar{3}$$

$$x_3 = 1.817263545$$

...

$$x_5 = x_6 = 1.817120593 \approx \sqrt[3]{6}$$

2) Solve $x^5 + x^4 - 5 = 0$

Solution $f(x) = x^5 + x^4 - 5 \Rightarrow f'(x) = 5x^4 + 4x^3 \Rightarrow x_{n+1} = x_n - \frac{x_n^5 + x_n^4 - 5}{5x_n^4 + 4x_n^3}$

1st guess: $x_1 = 1$

$$x_2 = 1 - \frac{1^5 + 1^4 - 5}{5 \cdot 1^4 + 4 \cdot 1^3} = 1.3$$

$$x_3 = 1.239420573$$

...

$$x_6 = x_7 = 1.224439550$$

3) Solve $x^3 + x = 1$

Solution $f(x) = x^3 + x - 1$ N.B. $\therefore$ Newton's Method only finds x-intercepts i.e. places where the function = 0.

$$\Rightarrow f'(x) = 3x^2 + 1 \Rightarrow x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}$$

$$x_1 = 1$$

$$x_2 = 1 - \frac{1^3 + 1 - 1}{3 \cdot 1^2 + 1} = 0.75$$

$$x_3 = 0.686046512$$

...

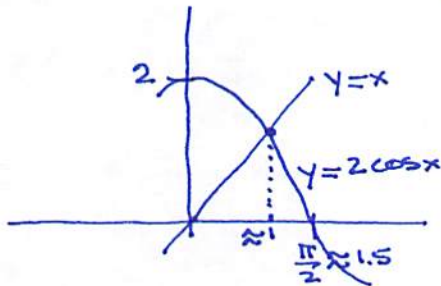
$$x_5 = x_6 = 0.682327804$$

4) Solve $2\cos x = x$

Solution $f(x) = 2\cos x - x$ ~~N.B.~~ N.B. Rearrange equation so solution, i.e. RHS, is 0.

$$\Rightarrow f'(x) = -2\sin x - 1 \Rightarrow x_{n+1} = x_n - \frac{2\cos x - x}{-2\sin x - 1}$$

How to find 1st guess? Consider graphs of $2\cos x$ and x ; our solution is where they cross:



$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1.03004337 \\ x_3 &= 1.02986654 \\ x_4 &= x_5 = 1.02986653 \end{aligned}$$

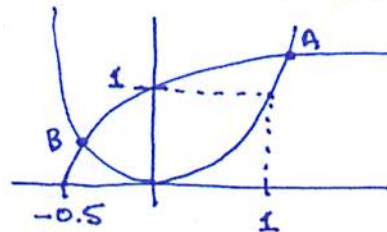
N.B. x is in radians so your calculator must be in radians mode.

5) Find x -coord. of intersection of $y = x^2$ and $y = \sqrt{2x+1}$

Sol'n $f(x) = x^2 - \sqrt{2x+1}$

$$\Rightarrow f'(x) = 2x - \frac{1}{\sqrt{2x+1}} = 2x - \frac{1}{\sqrt{2x+1}}$$

1st guesses via graphs:



For A: $x_1 = 1$

$$x_2 = 2$$

$$x_3 = 1.6\bar{3}$$

...

$$x_8 = x_9 = 1.395336994$$

For B: $x_1 = -0.5$

$$x_2 = -0.475$$

$$x_3 = -0.474626695$$

$$x_4 = x_5 = -0.474626618$$

6) Find the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$

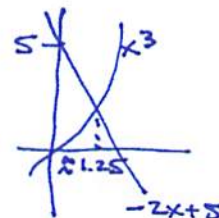
Solution Shape of f is roughly 

We are looking for its min value, i.e. here

At that point the tangent line to f is horizontal so $f'(x) = 0$
So we can restate the problem to be:

Solve $f'(x) = x^3 + 2x - 5 = 0 \Rightarrow f''(x) = 3x^2 + 2$

1st guess by seeing where graphs of x^3 and $-2x + 5$ cross:



$$x_1 = 1.25$$

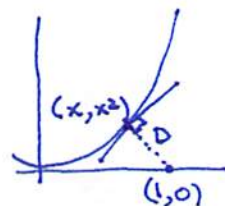
$$x_2 = 1.3317757009$$

...

$$x_4 = x_5 = 1.328268857$$

$\therefore$ the minimum value of f is $f(x_5) = -4.098859123$

7) Find coordinates of point on $y = x^2$ closest to $(1, 0)$.



Solution "closest to" $\Rightarrow$ minimize the distance D .

$$D = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} = \sqrt{(x^2 - 0)^2 + (x - 1)^2} = \sqrt{x^4 + x^2 - 2x + 1}$$

Easier to work with D^2 than D (and ok to $\because$ same value of x minimizes both D and D^2) so $f(x) = D^2 = x^4 + x^2 - 2x + 1$

Minimum D will occur at stationary point of D^2 i.e. when $f'(x) = 0$

$$f'(x) = 4x^3 + 2x - 2 = 0 \Rightarrow f''(x) = 12x^2 + 2$$

$$x_{n+1} = x_n - \frac{4x_n^3 + 2x_n - 2}{12x_n^2 + 2} = x_n - \frac{2x_n^3 + x_n - 1}{6x_n^2 + 1}$$

$x_1 = 1 \Rightarrow x_2 = 0.714285714 \Rightarrow \dots \Rightarrow x_6 = x_7 = 0.589754512$
 $y = f(x_6) = 0.347810385$ } coords of closest point
 eyeballed from graph above right.