

Homework Set 14: Solutions

1) $p(x) = 2x^3 - 3x^2 - 36x + 5$

y-int: $p(0) = 5$ (A)

x-int: $y \stackrel{!}{=} 0 = 2x^3 - 3x^2 - 36x + 5$ Trial and error gives $x=5$ as one root.
 $= (x-5)(\quad ? \quad)$

Use long division:

$$\begin{array}{r} 2x^2 + 7x - 1 \\ x-5 \overline{) 2x^3 - 3x^2 - 36x + 5} \\ \underline{2x^3 - 10x^2} \\ 7x^2 - 36x \\ \underline{7x^2 - 35x} \\ -x + 5 \\ \underline{-x + 5} \\ 0 \end{array}$$

$\Rightarrow 0 \stackrel{!}{=} (x-5)(2x^2 + 7x - 1)$

$x=5$ (B) $x = \frac{-7 \pm \sqrt{49 - 4(2)(-1)}}{2 \cdot 2} = \frac{-7 \pm \sqrt{57}}{4} \approx +0.14, -3.64$ (C) (D)

No symmetry.
 No vertical asymptotes.
 Horizontal asymptotes:

$\lim_{x \rightarrow +\infty} = +\infty = \lim_{x \rightarrow -\infty}$

Stationary points:

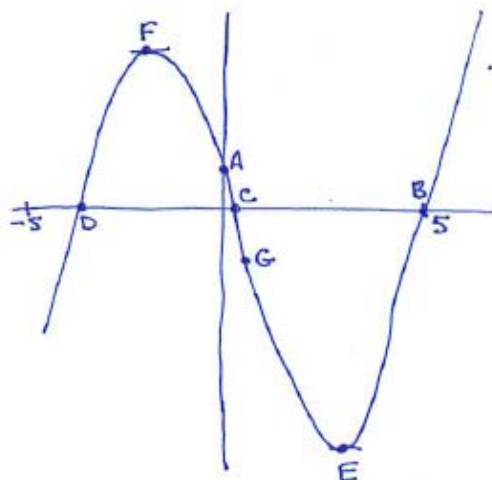
$p'(x) = 6x^2 - 6x - 36 \stackrel{!}{=} 0$
 $6(x-3)(x+2) = 0$

(E) $x=3$
 $y=-76$

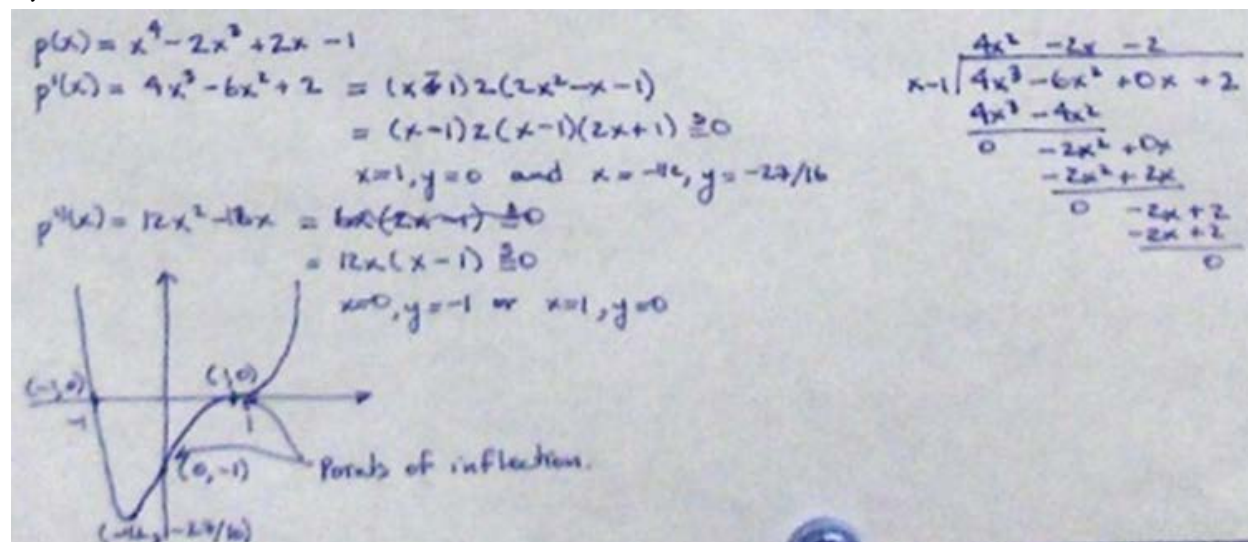
$x=-2$
 $y=49$ (F)

Points of inflection:

$p''(x) = 12x - 6 \stackrel{!}{=} 0$
 $x = 1/2$
 $y = -27/2$ (G)



2)



3)

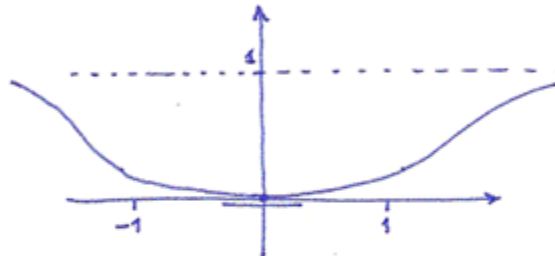
$$y = \frac{x^2}{x^2+4}$$

Note: always +ve $\therefore$ always above x-axis

x-int: $y \equiv 0 = \frac{x^2}{x^2+4}$, $x^2=0$, $x=0$

y-int: $y = \frac{0^2}{0^2+4} = 0$

symmetry: $\frac{x^2}{x^2+4} = \frac{(-x)^2}{(-x)^2+4}$ $\therefore$ symmetry about y-axis



vertical asymptotes: $x^2+4 \equiv 0$, $x^2 = -4$ no sol'n
 $\therefore$ no vert. asymptotes.

horizontal " : $\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2+4} = 1$

critical points: $y' = \frac{(x^2+4)2x - x^2(2x)}{(x^2+4)^2}$
 $= \frac{8x}{(x^2+4)^2} \equiv 0$ $x=0$

curvature: $y'' = \frac{(x^2+4)^2 \cdot 8 - 8x \cdot 2(x^2+4)2x}{(x^2+4)^4}$
 $= \frac{8x^2 + 32 - 32x^2}{(x^2+4)^3} \equiv 0$

$$-24x^2 + 32 = 0$$

$$x^2 = \frac{4}{3}, \quad x = \pm \frac{2}{\sqrt{3}}, \quad y = \frac{1}{4}$$

4)

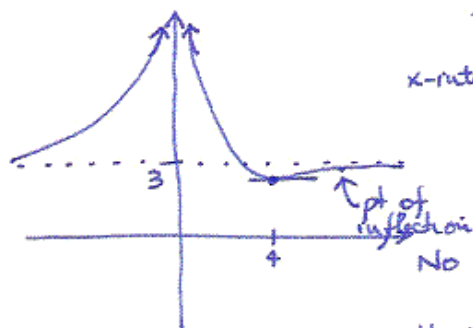
$$f(x) = \frac{4}{x^2} - \frac{2}{x} + 3$$

y-rntg
vertical asymptote @ $x=0$

$$\lim_{x \rightarrow 0^+} = +\infty, \quad \lim_{x \rightarrow 0^-} = +\infty$$

$$x\text{-int: } \frac{4}{x^2} - \frac{2}{x} + 3 \stackrel{?}{=} 0$$

$$4 - 2x + 3x^2 = 0 \quad x = \frac{2 \pm \sqrt{4 - 4 \cdot 3 \cdot 4}}{6} \quad (<0 \therefore \text{no soln})$$



No symmetries.

$$\text{Horiz. asymptote: } \lim_{x \rightarrow \infty} \frac{4}{x^2} - \frac{2}{x} + 3 = 3$$

$$\text{Critical points: } f'(x) = -8x^{-3} + 2x^{-2} \stackrel{?}{=} 0$$

$$\frac{-8}{x^3} + \frac{2}{x^2} = 0 \quad -8 + 2x = 0 \quad x = 4, y = \frac{11}{4} \quad (\text{N.B. } 3 = \frac{12}{4})$$

$$\text{Points of inflection: } f''(x) = 24x^{-4} - 4x^{-3} \stackrel{?}{=} 0$$

$$\frac{24}{x^4} - \frac{4}{x^3} = 0 \quad 24 - 4x = 0 \quad x = 6, y = \frac{25}{4} \quad (\text{N.B. } 3 = \frac{12}{4})$$

5)

Sketch $f(x) = \frac{x^2+x}{1-x^2} = \frac{x(x+1)}{(1-x)(1+x)} = \frac{x}{1-x}$ with hole at $x=-1, y=-\frac{1}{2}$

No symmetry.

Intercepts: $y = \frac{0}{1-0} = 0$, $0 = \frac{x}{1-x} \Rightarrow x=0$

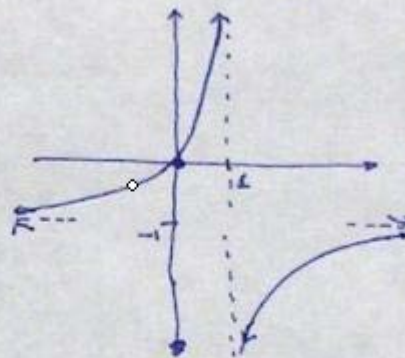
V. asymptote: $1-x \geq 0 \Rightarrow x=1$

$$\lim_{x \rightarrow 1^+} \frac{x}{1-x} = \frac{+1}{0^-} = -\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x}{1-x} = \frac{-1}{0^+} = +\infty$$

H. asymptote: $\lim_{x \rightarrow \pm\infty} \frac{x}{1-x} = \frac{\pm\infty}{-\infty} = -1$

$$\lim_{x \rightarrow -\infty} \frac{x}{1-x} = \frac{-\infty}{+\infty} = -1$$



Critical points: $f'(x) = \frac{(1-x) \cdot 1 - x(-1)}{(1-x)^2} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2} \geq 0$ No soln

$\therefore$ no stationary points.

No new other critical pts.

Pts of inflection: $f''(x) = \frac{-12(1-x)^{-3}(-1)}{(1-x)^4} = \frac{-2(1-x)}{(1-x)^4} = \frac{-2}{(1-x)^3} \geq 0$ No soln

Note that concavity matches diagram

$\therefore f''(x) > 0$ for $x < 1$ and (i.e. concave up) and
 $f''(x) < 0$ for $x > 1$ (i.e. concave down)

6)

Sketch $f(x) = \frac{x^2-2}{x} = x - \frac{2}{x} \quad \therefore y=x$ is an oblique asymptote

Symmetry: None

Intercepts: $y = -\frac{2}{0}$ DNE

$$0 = \frac{x^2-2}{x} \Rightarrow x = \pm\sqrt{2}$$

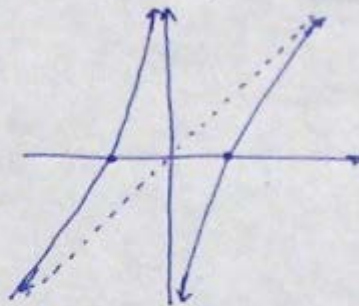
V. asymptote: $\lim_{x \rightarrow 0^+} \frac{-}{0^+} = -\infty$

$$\lim_{x \rightarrow 0^-} \frac{-}{0^-} = +\infty$$

H. asymptote: None, but see oblique above $\lim_{x \rightarrow \infty} = +\infty, \lim_{x \rightarrow -\infty} = -\infty$

Critical points: $f'(x) = 1 + \frac{2}{x^2} \stackrel{?}{=} 0 \quad x^2 + 2 = 0 \quad x = \pm\sqrt{-2}$ DNE, and no new crit. pts.

Pts of inflection: $f''(x) = -\frac{4}{x^3} \stackrel{?}{=} 0 \Rightarrow -4 = 0$ No sol'n



7)

Sketch $f(x) = \frac{(x-2)^3}{x^2} = x - 6 + \frac{12x-8}{x^2} \quad \therefore y=x-6$ is an oblique asymptote

Symmetry: None

Intercepts: $y = -\frac{8}{0}$ DNE

$$0 = \frac{(x-2)^3}{x^2} \Rightarrow x-2=0 \Rightarrow x=2$$

V. asymptote: $\lim_{x \rightarrow 0^+} \frac{-}{0^+} = -\infty$

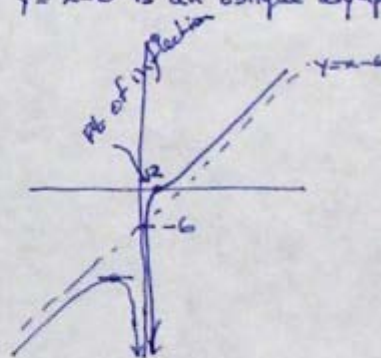
$$\lim_{x \rightarrow 0^-} \frac{-}{0^-} = -\infty$$

Critical points: $f'(x) = 1 + \frac{12x^2 - 24x + 16}{x^3} = 1 + \frac{16x^2 - 12x^3}{x^3}$
 $= \frac{x^3 - 12x + 16}{x^3} = \frac{(x+4)(x-2)^2}{x^3} \stackrel{?}{=} 0 \quad x = -4, x = 2$
 $y = -\frac{28}{4} \quad y = 0$

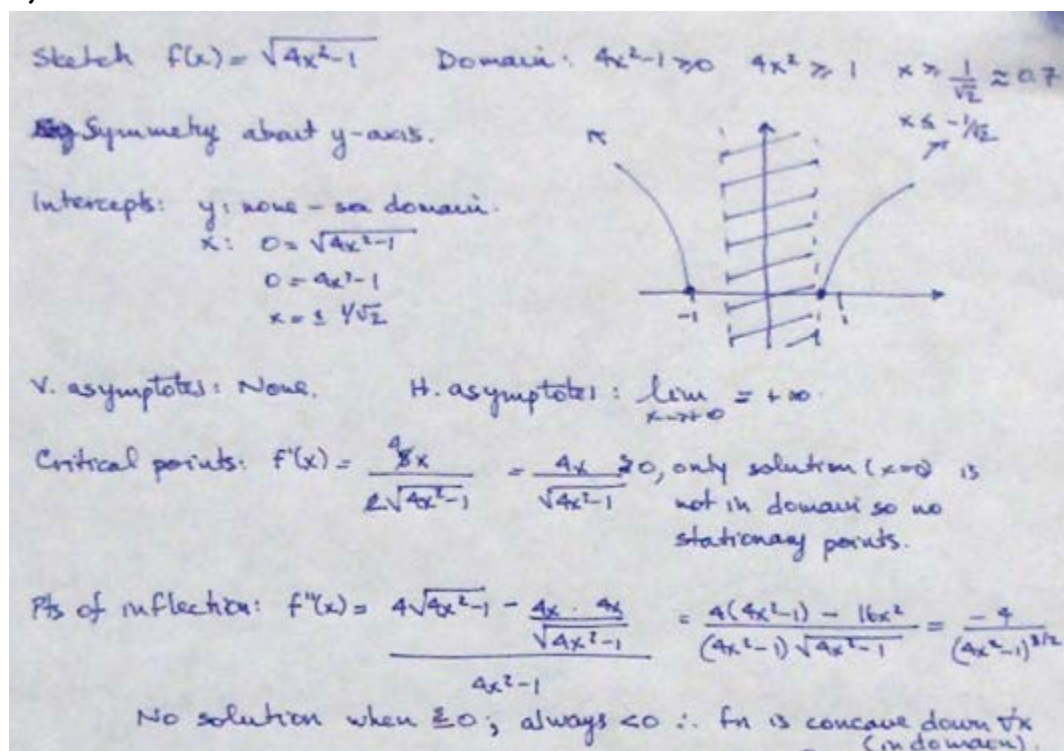
Pts of inflection: $f''(x) = \frac{(x+4)2(x-2) + (x-2)^2}{x^4} = \frac{(x+4)(x-2)^2}{x^4}$

$\therefore$ one at $x=2, y=0$

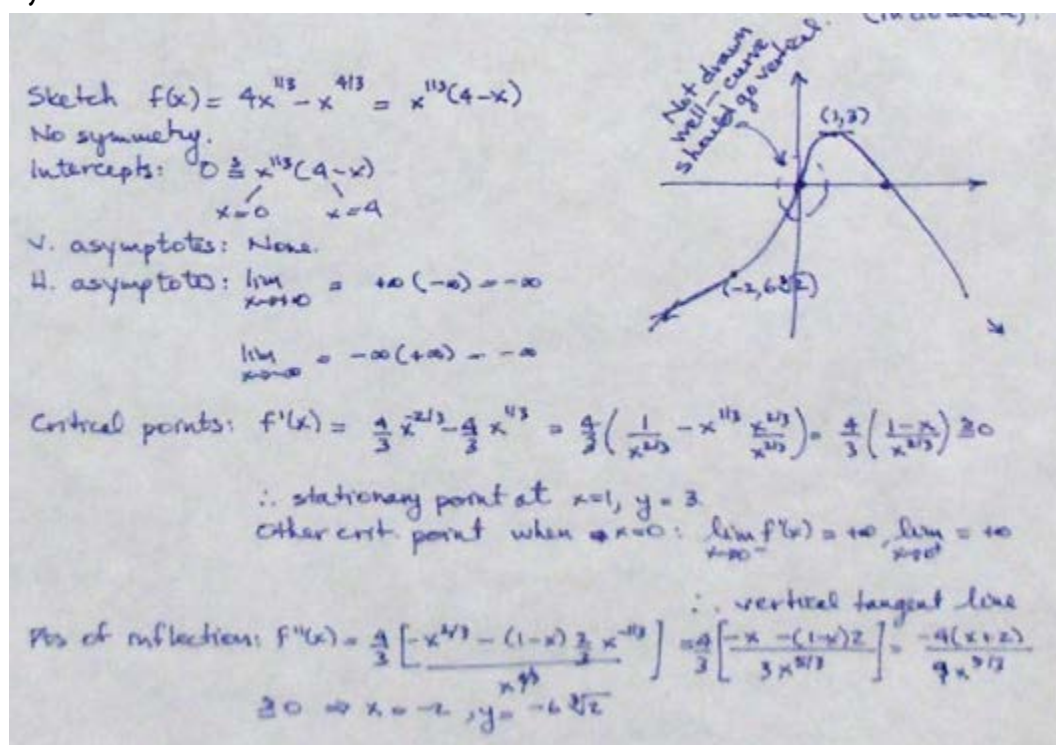
$$f''(x) = \frac{(x+4)(x-2)^2}{x^4} = \frac{(x+4)(x^2-4x+4)}{x^4} = \frac{x^3+4x^2-4x^2-16x+4x^2-16x-24}{x^4} = \frac{24(x-2)}{x^4}$$



8)



9)



10)

Sketch $f(x) = \frac{8+x}{2+\sqrt[3]{x}}$ Note: $x^{1/3} = \sqrt[3]{x}$

$$= x^{2/3} - 2x^{1/3} + 4$$

Intercepts: $x=0 \Rightarrow y=4$
 $y=0 \Rightarrow \frac{8+x}{2+\sqrt[3]{x}} = 0$
 $8+x=0$
 $x=-8$ but at $x=-8$ $f(x)$ DNE $\therefore$ has form $\frac{0}{0}$!
 $(\therefore$ no x-ruts).

V. asymptote? $2+\sqrt[3]{x} \leq 0 \Rightarrow \sqrt[3]{x} = -2 \Rightarrow x = -8 \therefore$ No asymptote, but hole $P(-8, 12)$

H. asymptote? $\lim_{x \rightarrow \infty} = +\infty \quad \lim_{x \rightarrow -\infty} = +\infty$

Crit. pts: $f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3}$
 $= \frac{2}{3} \left(\frac{1}{x^{1/3}} - \frac{1}{x^{2/3}} \right) = \frac{2}{3} \frac{x^{1/3} - 1}{x^{4/3}}$
 $\geq 0 \Rightarrow x^{1/3} - 1 = 0 \Rightarrow x^{1/3} = 1 \Rightarrow x = 1, y = 3$

Other crit pt: $x^{1/3} = 0$ or $x = 0$. $\lim_{x \rightarrow 0^+} f'(x) = +\infty, \lim_{x \rightarrow 0^-} f'(x) = -\infty$
 $\therefore$ vertical tangent line

Pts of inflection: $f''(x) = \frac{2}{3} \cdot \frac{-1}{3} x^{-4/3} - \frac{2}{3} \left(\frac{-2}{3} \right) x^{-5/3}$
 $= \frac{-2}{9} \left(\frac{1}{x^{4/3}} - \frac{2}{x^{5/3}} \right) = \frac{-2(x^{1/3} - 2)}{9x^{5/3}} \geq 0 \Rightarrow x^{1/3} - 2 = 0$
 $x^{1/3} = 2$
 $x = 8$
 $y = 4$

11)

Sketch $y = x + \sin x$

$y' = 1 + \cos x \geq 0 \Rightarrow \cos x = -1$
 $x = \pi \pm 2n\pi$

$y'' = -\sin x \geq 0 \Rightarrow x = n\pi$
 $n = 0, 2, 4, \dots$