

Homework Set 15: Solutions

1. "the sum of the number and its reciprocal"
$$x + \frac{1}{x} = f(x)$$

a) Minimize $f(x)$ on $[\frac{1}{2}, \frac{3}{2}]$

Solution Note that $f(x)$ is continuous on $[\frac{1}{2}, \frac{3}{2}]$, so we check:

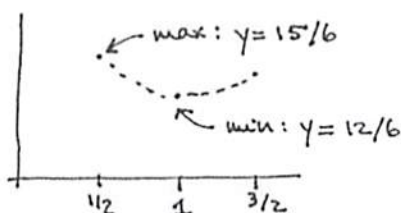
Stationary points: $f'(x) = 1 - \frac{1}{x^2} \stackrel{!}{=} 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$
but only $+1$ is in $[\frac{1}{2}, \frac{3}{2}]$

at $x=1$, $f(1) = \frac{1}{2} + 1 = \frac{3}{2} (= 12/6)$

Other critical points: $f'(x)$ DNE @ $x=0$ but $f(0)$ also DNE
and 0 is not in $[\frac{1}{2}, \frac{3}{2}]$

End points: $f(\frac{1}{2}) = \frac{1}{2} + \frac{1}{1/2} = \frac{1}{2} + 2 = \frac{5}{2} (= 15/6)$

$f(\frac{3}{2}) = \frac{3}{2} + \frac{1}{3/2} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$

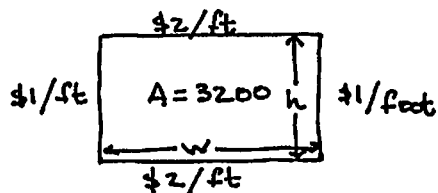


$\therefore f$ is minimized when $x=1$

b) Maximize $f(x)$ $[\frac{1}{2}, \frac{3}{2}]$

Solution Consider same points as above. We see that f is maximized when $x = \frac{1}{2}$.

2. Given



Minimize Cost

Solution Cost = $2 \cdot 2w + 1 \cdot 2h = 4w + 2h$

Need to eliminate either w or h before differentiating.

Use: Area = $3200 = w \cdot h \Rightarrow w = 3200/h$

$\therefore \text{Cost} = 4 \cdot \frac{3200}{h} + 2h = \frac{12800}{h} + 2h$

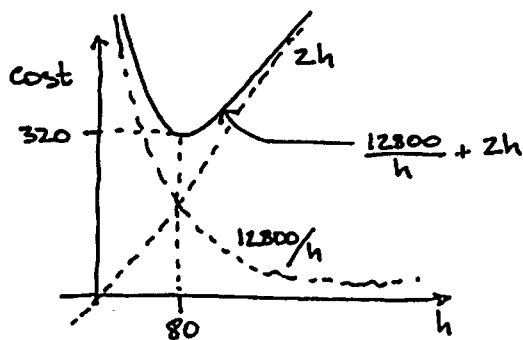
$\frac{d\text{Cost}}{dh} = -\frac{12800}{h^2} + 2 \stackrel{!}{=} 0 \Rightarrow 2h^2 = 12800 \Rightarrow h = \pm \sqrt{6400} = \pm 80$

@ $h=80$: Cost = $\frac{12800}{80} + 2 \cdot 80 = 320$ and $w = 3200/80 = 40$

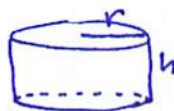
Check endpoints: Minimum value of h is 0, Cost(0) = undefined

Maximum value of h is ∞ , $\lim_{h \rightarrow \infty} \text{Cost} = +\infty$

$\therefore$ Minimum cost is \$320 achieved with $h=80$ and $w=40$.



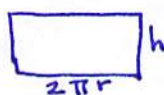
3. Minimize material in open can



Subject to $V = 500 \text{ cm}^3$

Solution material = bottom + sides

$$= \pi r^2 + 2\pi r h$$



$$\therefore \text{Minimize } S = \pi r^2 + 2\pi r h \quad \text{subject to } V = 500 = \pi r^2 h$$
$$\frac{500}{\pi r^2} = h$$

$$S = \pi r^2 + 2\pi r \cdot \frac{500}{\pi r^2}$$

$$= \pi r^2 + \frac{1000}{r}$$

Stationary
points

$$\frac{dS}{dr} = 2\pi r - \frac{1000}{r^2} \stackrel{!}{=} 0$$

$$2\pi r^3 - 1000 = 0$$

$$r = \sqrt[3]{\frac{1000}{2\pi}} \Rightarrow S = \pi \left(\frac{1000}{2\pi} \right)^{2/3} + \frac{1000}{\left(\frac{1000}{2\pi} \right)^{1/3}}$$

$$\approx 276.79$$

Other crit.
points

$r^2 = 0 \Rightarrow r = 0$ but as $r \rightarrow 0$ $S \rightarrow \infty$ so not a minimum.

Endpoints

0: see above $\nearrow$

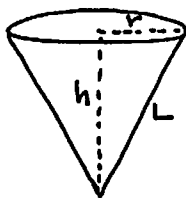
$\rightarrow \infty$: $S \rightarrow \infty$ $\therefore$ not a minimum

$\therefore$ dimensions to minimize material are $r = \sqrt[3]{\frac{1000}{2\pi}}$

$$\text{and } h = \frac{500}{\pi \left(\sqrt[3]{\frac{1000}{2\pi}} \right)^2}$$

and the amount of material required will be $\approx 277 \text{ cm}^2$

4. Given



$$V = 100 \text{ cm}^3 = \frac{1}{3} \pi r^2 h$$

$$\Rightarrow h = \frac{300}{\pi r^2}$$

Minimize "the amount of paper" = surface area of the cup
 $= S = \pi r L$ (from front cover of text)
 $= \pi r \sqrt{r^2 + h^2}$

$$= \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2}$$

This looks miserable to differentiate but we can make it easier $\therefore$ whatever value of r minimizes S will also minimize S^2 .

So switch gears to minimizing $A = S^2 = \pi^2 r^2 \left(r^2 + \left(\frac{300}{\pi r^2}\right)^2 \right)$
 $= \pi^2 r^4 + \frac{90000}{r^2}$

$$\frac{dA}{dr} = 4\pi^2 r^3 - \frac{180000}{r^3} = \frac{4(\pi^2 r^6 - 45000)}{r^3}$$

Stationary point(s): $\frac{dA}{dr} \stackrel{!}{=} 0 = \frac{4(\pi^2 r^6 - 45000)}{r^3}$

$$0 = \pi^2 r^6 - 45000$$

$$r = \sqrt[6]{45000/\pi^2} = \sqrt{2} \sqrt[3]{75/\pi}$$

Other critical point(s): $r=0$, but if $r=0$ V cannot be 100.

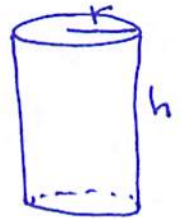
End points: As $r \rightarrow 0$, $A = \pi^2 r^4 + \frac{90000}{r^2} \rightarrow \infty$

As $r \rightarrow \infty$, $A = \pi^2 r^4 + \frac{90000}{r^2} \rightarrow \infty$

$\therefore$ Minimum amount of paper is used when $r = \sqrt{2} \sqrt[3]{75/\pi}$
 $\approx 4.07 \text{ cm}$

and $h = \frac{300}{\pi (\sqrt{2} \sqrt[3]{75/\pi})^2} = 2 \sqrt[3]{75/\pi} \approx 5.76 \text{ cm}$

5. Minimize cost $C = \underbrace{2 \cdot 2 \cdot \pi r^2}_{\text{cost of 2 ends}} + \underbrace{1 \cdot 2 \pi r h}_{\text{cost of sides}}$



subject to $V = 48\pi \text{ cm}^3 = \pi r^2 h \Rightarrow h = 48\pi / \pi r^2$

Solution $C = 6\pi r^2 + 2\pi r \cdot \frac{48\pi}{\pi r^2}$
 $= 6\pi r^2 + \frac{96\pi}{r}$

$$\frac{dC}{dr} = 12\pi r - \frac{96\pi}{r^2}$$

Stationary points $\frac{dC}{dr} \stackrel{!}{=} 0 = 12\pi r - \frac{96\pi}{r^2}$

$$0 = 12r^3 - 96$$

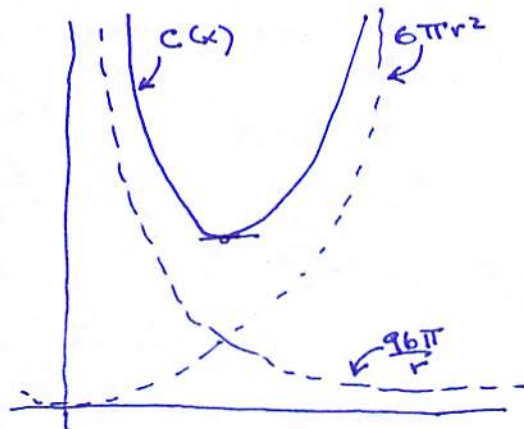
$$r = \sqrt[3]{\frac{96}{12}} = \sqrt[3]{8} = 2 \Rightarrow C = 6\pi 2^2 + \frac{96\pi}{2} = 72\pi$$

Other crit. points $\frac{dC}{dr}$ DNE @ $r=0$ but as $r \rightarrow 0$, $C \rightarrow \infty$ $\therefore$ not a min

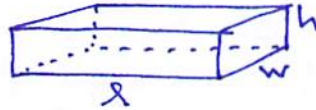
End points $r=0$ see $\nearrow$

$r \rightarrow \infty$: $C \rightarrow \infty$ so not a min.

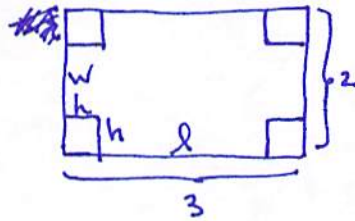
For interest: Shape of $C(r)$:



6. Maximize $V = l \cdot w \cdot h$



Subject to



$$\begin{aligned} \therefore w + 2h &= 2 \Rightarrow w = 2 - 2h \\ l + 2h &= 3 \Rightarrow l = 3 - 2h \end{aligned}$$

Solution

$$\begin{aligned} V &= l \cdot w \cdot h = (3 - 2h)(2 - 2h)h \\ &= (6 - 10h + 4h^2)h \\ &= 6h - 10h^2 + 4h^3 \end{aligned}$$

Stationary points:

$$\begin{aligned} \frac{dV}{dh} &= 6 - 20h + 12h^2 \stackrel{!}{=} 0 \Rightarrow h = \frac{20 \pm \sqrt{(-20)^2 - 4(12)6}}{2 \cdot 12} \\ &= \frac{20 \pm \sqrt{112}}{24} \\ &\approx 1.274 \text{ \& } 0.392 \end{aligned}$$

Now max possible value of h is 1 so $\nearrow$ is not possible.

$$h \approx 0.392 \Rightarrow V \approx 1.056 \text{ m}^3$$

End points:

$$\begin{aligned} h=0 &\Rightarrow V=0 \\ h=1 &\Rightarrow V=0 \quad (\because w \text{ will be } 0) \end{aligned}$$

Other crit. points: None.

$\therefore$ Optimal volume of $\approx 1.056 \text{ m}^3$ is achieved when 0.392×0.392 squares are cut from each corner.

7. Find proportions of can to minimize material.

Solution Volume not specified so just call it V .

Minimize $S = 2\pi r^2 + 2\pi rh$ where $V = \pi r^2 h \Rightarrow h = V/\pi r^2$

$$\therefore S = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2} \stackrel{!}{=} 0 \Rightarrow 4\pi r^3 - 2V = 0$$

$$r = \sqrt[3]{\frac{2V}{4\pi}} = \sqrt[3]{\frac{V}{2\pi}}$$

$$h = \frac{V}{\pi \left(\sqrt[3]{\frac{V}{2\pi}}\right)^2}$$

$$= \frac{V}{\pi \frac{V^{2/3}}{2^{2/3} \pi^{2/3}}}$$

$$= \frac{V^{1/3}}{\frac{\pi^{1/3}}{2^{2/3}}}$$

$$= \sqrt[3]{\frac{4V}{\pi}}$$

$$2^{2/3} = (2^2)^{1/3} = 4^{1/3}$$

More interestingly, regardless of the volume the ratio of the height to the radius should be:

$$\begin{aligned} \frac{h}{r} &= \frac{\sqrt[3]{\frac{4V}{\pi}}}{\sqrt[3]{\frac{V}{2\pi}}} \\ &= \sqrt[3]{\frac{2 \frac{4V}{\pi}}{\frac{V}{2\pi}}} \\ &= \sqrt[3]{2} \end{aligned}$$

$$\approx 1.26$$

So the height should be ≈ 1.26 times the radius.

8. a) $70 = C(0)$ = the fixed cost of producing the paper i.e. the amount we have to pay even if we print no papers, e.g. cost of the building, fixed salaries.

b) Revenue: $R(x) = \$1 \cdot x = \$x$

c) Profit = Revenue - Cost

$$= R(x) - C(x)$$

$$P(x) = x - (70 + 0.001x + 0.00005x^2)$$

d) Maximize $P(x) = 0.999x - 70 - 0.00005x^2$

$$P'(x) = 0.999 - 0 - 2 \cdot 0.00005x$$

$$= 0.999 - 0.0001x \stackrel{!}{=} 0$$

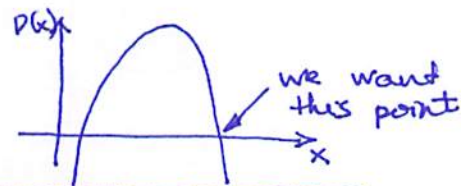
$$x = \frac{0.999}{0.0001}$$

$$= 9,990$$

At this point the marginal profit, $P'(x)$, is 0.
 " " " " " revenue, $R'(x)$, is 1.

e) Maximize x subject to $P(x) \geq 0$

$P(x)$ is a parabola opening downward



Its x intercepts are
$$x = \frac{-0.999 \pm \sqrt{0.999^2 - 4(-0.00005)(-70)}}{2 \cdot (-0.00005)}$$

$$= \frac{-0.999 \pm \sqrt{0.984001}}{-0.0001}$$

$$\approx 19,909, 70$$

Their optimum would be to produce 19,909 copies.

The marginal profit would be $P'(x) = -0.9919$

The marginal revenue would remain 1.

9. a) Given Cost = $C(x) = 100\,000 + 50x + 0.0025x^2$
 x in $[0, 7,000]$
 Revenue = $R(x) = 100x$

Maximize Profit = $P(x) = R(x) - C(x) = 100x - (100\,000 + 50x + 0.0025x^2)$
 $= 50x - 100\,000 - 0.0025x^2$

Stationary point(s) $P'(x) = 50 - 0.005x \stackrel{!}{=} 0$
 $x = 50/0.005$
 $= 10,000$ but this is outside range so not possible.

Other crit. points None ($\because$ polynomial)

Endpoints $0: P(0) = 50 \cdot 0 - 100\,000 - 0.0025 \cdot 0^2$
 $= -100\,000$

So if we produce no acid we lose \$100,000/day due to our fixed costs.

$7,000: P(7,000) = 50(7,000) - 100\,000 - 0.0025(7,000)^2$
 $= \$127,500$

$\therefore$ To maximize profit (a net \$127,500) we should manufacture 7,000 units

b) $P'(7,000) = 50 - 0.005(7,000) = 15$ So our profit would increase $\approx \$15$ if we could increase production to 7,001 units from 7,000 units.

c) Yes we should increase capacity as long as $P'(x) > 0$ i.e. until $P'(x) = 0$ or until we are manufacturing 10,000 units

10. Given $C(r) = 40 + 10r^2 = \text{Cost}$

a) Find $C'(10)$ Solution $C'(r) = 20r \quad \therefore C'(10) = 20 \cdot 10 = 200$

b) Given Subsidy $= S(r) = 500r$

How will the firm respond

Solution Assuming the firm will seek to maintain its cost at

$C(10) = 40 + 10 \cdot 10^2 = 4,000$ with the revised cost formula:

$C(r) = 40 + 10r^2 - 500r \stackrel{!}{=} 4000 \Rightarrow r^2 - 50r - 396 = 0$

$$r = \frac{50 \pm \sqrt{(-50)^2 - 4(1)(-396)}}{2 \cdot 1}$$

$r \approx 56.95 \rightarrow 6.95$

It should be willing to achieve a reduction of ≈ 57 lbs.

11. Maximize total yield $Y = \underbrace{\text{yield per tree}}_y \times \underbrace{\# \text{ of trees}}_t$

Given "yield per tree depends linearly on # of trees"

and $t = 50 \Rightarrow Y = 75$

$t = 40 \Rightarrow Y = 80$

Solution $t = 50, Y = 75 \Rightarrow \text{yield per tree} = y = 75/50 = 1.5 \text{ bags/tree}$
 $t = 40, Y = 80 \Rightarrow \text{yield per tree} = y = 80/40 = 2.0 \text{ bags/tree}$

y as linear function of $t \Rightarrow y = mt + b \quad m = \frac{y_2 - y_1}{t_2 - t_1} = \frac{2 - 1.5}{40 - 50} = -0.05$

$2 = -0.05(40) + b \Rightarrow b = 4$

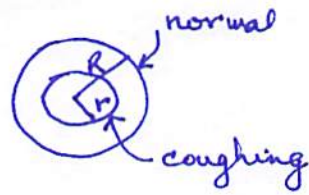
$\therefore y = -0.05t + 4$

Sub in $Y = yt = (-0.05t + 4) \cdot t = -0.05t^2 + 4t$

$Y' = -0.1t + 4 \stackrel{!}{=} 0 \quad t = 40$

So the current 40 trees is in fact the optimal solution

12. Given Trachea cross-section



a) constriction = $R - r$

b) $v \propto P \propto R - r \quad \therefore v = k_1(R - r)$

c) $A = \pi r^2$

d) $m \propto A \quad \therefore m = k_2 A = k_2 \pi r^2$

e) $M = \cancel{m} m v = k_2 \pi r^2 \cdot k_1(R - r) = k_1 k_2 \pi r^2 (R - r) = k_1 k_2 \pi R r^2 - k_1 k_2 \pi r^3$

f) $\frac{dM}{dr} = 2k_1 k_2 \pi R r - 3k_1 k_2 \pi r^2 \stackrel{!}{=} 0$

$$\begin{array}{l} r(2R - 3r) = 0 \\ \swarrow \quad \searrow \\ r = 0 \quad 2R - 3r = 0 \\ \quad \quad r = \frac{2R}{3} \end{array}$$

g) We should constrict our trachea ^{radius} by $\frac{1}{3}$