

Homework Set 17: Integration by Substitution

Evaluate the following integrals:

1. $\int \frac{(4x-5)^9 dx}{u \frac{du}{4}}$ $u = 4x-5 \Rightarrow \frac{du}{dx} = 4 \Rightarrow dx = \frac{du}{4} \Rightarrow \int u^9 \frac{du}{4} = \frac{1}{4} \cdot \frac{1}{10} u^{10} + C = \frac{(4x-5)^{10}}{40} + C$
2. $\int \frac{\sin 5x dx}{u \frac{du}{5}}$ $u = 5x \Rightarrow du = 5 dx \Rightarrow dx = \frac{du}{5} \Rightarrow \int \sin u \cdot \frac{du}{5} \Rightarrow -\frac{1}{5} \cos u + C = -\frac{\cos 5x}{5} + C$
3. $\int \sec^2 7x dx$ $u = 7x \Rightarrow dx = \frac{du}{7} \Rightarrow \frac{1}{7} \int \sec^2 u du = \frac{1}{7} \tan u + C = \frac{1}{7} \tan 7x + C$
4. $\int \theta \sqrt{5\theta^2+3} d\theta$ $u = 5\theta^2+3 \Rightarrow \frac{du}{d\theta} = 10\theta \Rightarrow \frac{du}{10} = \theta d\theta \Rightarrow \frac{1}{10} \int \sqrt{u} du = \frac{1}{10} \cdot \frac{2}{3} u^{3/2} + C = \frac{(5\theta^2+3)^{3/2}}{15} + C$
5. $\int \frac{x^3}{(5x^4+2)^2} dx$ $u = 5x^4+2 \Rightarrow \frac{du}{dx} = 20x^3 \Rightarrow x^3 dx = \frac{du}{20} \Rightarrow \frac{1}{20} \int \frac{1}{u^2} du = -\frac{1}{20u} + C = -\frac{1}{20(5x^4+2)} + C$
6. $\int \frac{\cos x}{\sqrt{\sin x-2}} dx$ $u = \sin x - 2 \Rightarrow \frac{du}{dx} = \cos x \Rightarrow \int \frac{1}{\sqrt{u}} du \Rightarrow \frac{2}{3} u^{3/2} + C \Rightarrow \frac{2}{3} (\sin x - 2)^{3/2} + C$
7. $\int \frac{\cos(5/x)}{x^2} dx$ $u = 5/x \Rightarrow \frac{du}{dx} = -\frac{5}{x^2} \Rightarrow -\frac{1}{5} \int \cos u du = -\frac{1}{5} \sin u + C = -\frac{1}{5} \sin\left(\frac{5}{x}\right) + C$
8. $\int \sin 2t \cos^4 2t dt$ $u = \cos 2t \Rightarrow \frac{du}{dt} = -2 \sin 2t \Rightarrow -\frac{1}{2} \int u^4 du = -\frac{1}{10} u^5 + C = -\frac{\cos^5 2t}{10} + C$
9. $\int x \sec^2(x^2) dx$ $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{1}{2} \int \sec^2 u du = \frac{1}{2} \tan u + C = \frac{1}{2} \tan x^2 + C$
10. $\int \sin 5x \sqrt{1-\sin 5x} dx$ $u = 1 - \sin 5x \Rightarrow \frac{du}{dx} = -\cos 5x \Rightarrow -\frac{1}{5} \int \sqrt{u} du = -\frac{1}{5} \cdot \frac{2}{3} u^{3/2} + C = -\frac{2(1-\sin 5x)^{3/2}}{15} + C$
11. $\int \sec^2 x \tan x dx$ $u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x \Rightarrow \int u du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$
12. $\int \sec^3 x \tan x dx$ $u = \sec x \Rightarrow \frac{du}{dx} = \sec x \tan x \Rightarrow \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$
13. $\int x \sqrt{2-x} dx$ $u = 2-x \Rightarrow x = 2-u \Rightarrow \int (2-u) \sqrt{u} du = -\int (2-u^{1/2}) du = -2 \cdot \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2} + C = -\frac{4}{3} (2-x)^{3/2} + \frac{2}{5} (2-x)^{5/2} + C$
14. $\int \tan^3 x \sec^2 x dx$ $u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x \Rightarrow \int u^3 du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$
15. $\int \tan^3 x \sec x dx = \int \tan^2 x \cdot \sec x \tan x dx = \int (\sec^2 x - 1) \sec x \tan x dx$
 $u = \sec x \Rightarrow du = \sec x \tan x dx$
 $= \int (u^2 - 1) du = \frac{u^3}{3} - u + C = \frac{\sec^3 x}{3} - \sec x + C$

$$u = \tan 3x \Rightarrow \frac{du}{dx} = 3 \sec^2 3x \Rightarrow \frac{1}{3} \int (u^2 + 1) du = \frac{1}{3} \left(\frac{u^3}{3} + u \right) + C$$

$$= \frac{1}{3} \left(\frac{\tan^3 3x}{3} + \tan 3x \right) + C$$

$$16. \int \sec^4 3x dx = \int \sec^2 3x \cdot \sec^2 3x dx = \int (\tan^2 3x + 1) \sec^2 3x dx$$

Evaluate each of the following integrals assuming that n is positive and $b \neq 0$:

$$17. \int (a+bx)^n dx \quad u = a+bx \Rightarrow \frac{du}{dx} = b \Rightarrow \frac{1}{b} \int u^n dx = \frac{1}{b} \frac{1}{n+1} u^{n+1} + C = \frac{(a+bx)^{n+1}}{b(n+1)} + C$$

$$18. \int \sqrt[n]{a+bx} dx = \int (a+bx)^{1/n} dx \quad u = a+bx \Rightarrow \frac{du}{dx} = b \Rightarrow \frac{1}{b} \int u^{1/n} du = \frac{1}{b} \frac{n}{n+1} u^{n+1/n} + C = \frac{n}{b(n+1)} (a+bx)^{1+1/n} + C$$

$$19. \int \sin^n(a+bx) \cos(a+bx) dx \quad u = \sin(a+bx) \Rightarrow \frac{du}{dx} = \cos(a+bx) \cdot b \Rightarrow \frac{1}{b} \int u^n du = \frac{1}{b} \frac{1}{n+1} u^{n+1} + C = \frac{\sin^{n+1}(a+bx)}{b(n+1)} + C$$

20. Evaluate $\int \sin x \cos x dx$ (i) using the substitution $u = \sin x$ and (ii) using the substitution $u = \cos x$. Are these actually two different answers or are they equivalent?

$$(i) = \int u du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C \quad (ii) = -\int u du = -\frac{u^2}{2} + C = -\frac{\cos^2 x}{2} + C$$

(i) $\frac{\sin^2 x}{2} + C = \frac{1 - \cos^2 x}{2} + C = \frac{1}{2} - \frac{\cos^2 x}{2} + C = -\frac{\cos^2 x}{2} + C - \frac{1}{2}$ which differs from (ii) only by the constant $-\frac{1}{2}$ but since C 's arbitrary this difference is irrelevant.

21. Evaluate $\int (5x-1)^2 dx$ (i) using the substitution $u = 5x-1$ and (ii) by expanding the expression and integrating the resulting polynomial. Are these actually two different answers or are they equivalent?

$$(i) u = 5x-1 \quad \frac{du}{dx} = 5 \Rightarrow \frac{1}{5} \int u^2 du = \frac{u^3}{15} + C = \frac{(5x-1)^3}{15} + C$$

$$(ii) \int (5x-1)^2 dx = \int (25x^2 - 10x + 1) dx = \frac{25x^3}{3} - \frac{10x^2}{2} + x + C = \frac{25x^3}{3} - 5x^2 + x + C$$

$$\text{Comparing: } = \frac{(25x^2 - 10x + 1)(5x-1)}{15} + C$$

$$= \frac{125x^3 - 50x^2 + 5x - 25x^2 + 10x - 1}{15} + C$$

$$= \frac{125x^3 - 75x^2 + 15x - 1}{15} + C = \frac{25}{3}x^3 - 5x^2 + x - \frac{1}{15} + C$$

which is equal to (the $-\frac{1}{15}$ difference is irrelevant of C)