

Homework Set 19: Evaluating Definite Integrals by Substitution

Evaluate the following integrals:

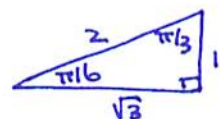
$$1. \int_0^2 (3x^2 - 64x + 8) dx = \left[x^3 - 32x^2 + 8x \right]_0^2 = (2^3 - 32 \cdot 2^2 + 8 \cdot 2) - 0 = -104$$

$$2. \int_1^2 \frac{3}{x^5} dx = 3 \int_1^2 x^{-5} dx = -\frac{3}{4} x^{-4} \Big|_1^2 = -\frac{3}{4} \left[\frac{1}{2^4} - \frac{1}{1^4} \right] = -\frac{3}{4} \left[-\frac{15}{16} \right] = \frac{45}{64}$$

$$3. \int_4^9 \frac{2}{x\sqrt{x}} dx = 2 \int_4^9 \frac{1}{x^{3/2}} dx = 2 \int_4^9 x^{-3/2} dx = 2 \left[-2x^{-1/2} \right]_4^9 = -4 \left[\frac{1}{\sqrt{9}} - \frac{1}{\sqrt{4}} \right] = -4 \left[-\frac{1}{6} \right] = \frac{2}{3}$$

$$4. \int_0^{\pi/4} \sin x dx = -\cos x \Big|_0^{\pi/4} = -\cos \frac{\pi}{4} - (-\cos 0) = -\frac{1}{\sqrt{2}} + 1 = 1 - \frac{1}{\sqrt{2}}$$

$$5. \int_{\pi/6}^{\pi/3} \csc^2 \theta d\theta = -\cot \theta \Big|_{\pi/6}^{\pi/3} = -\cot \frac{\pi}{3} - (-\cot \frac{\pi}{6}) = -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{3-1}{\sqrt{3}}$$



Typo! -----

$$6. \int_0^{\pi/6} \tan^2 x - \sec x \tan x dx = \int_0^{\pi/6} (\sec^2 x - 1) - \sec x \tan x dx = \left[\tan x - x - \sec x \right]_0^{\pi/6} = \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6} - \frac{2}{\sqrt{3}} \right) - (0 - 0 - 1) = 1 - \frac{1}{\sqrt{3}} - \frac{\pi}{6}$$

$$7. \int_1^2 \sqrt{3x-1} dx = \frac{1}{3} \frac{2}{3} (3x-1)^{3/2} \Big|_1^2 = \frac{2}{9} \left[5^{3/2} - 2^{3/2} \right]$$

$$8. \int_{\pi/6}^{\pi/3} \sin x (\cos x + 1)^5 dx \quad \left. \begin{array}{l} u = \cos x \\ du = -\sin x dx \end{array} \right\} - \int_{\sqrt{3}/2}^{1/2} (u+1)^5 du = -\frac{1}{6} (u+1)^6 \Big|_{\sqrt{3}/2}^{1/2} = -\frac{1}{6} \left[\left(\frac{3+\sqrt{3}}{2} \right)^6 - \left(\frac{3}{2} \right)^6 \right]$$

$$9. \int_0^1 \frac{dx}{x^2 - 4x + 4} = \int_0^1 \frac{dx}{(x-2)^2} \quad \left. \begin{array}{l} u = x-2 \\ du = dx \\ x=0 \Rightarrow u = -2 \\ x=1 \Rightarrow u = -1 \end{array} \right\} = \int_{-2}^{-1} u^{-2} du = -u^{-1} \Big|_{-2}^{-1} = -\frac{1}{-1} - (-\frac{1}{-2}) = \frac{1}{2}$$

$$10. \int_0^{\pi/4} \sqrt{\tan x} \sec^2 x dx \quad \left. \begin{array}{l} u = \tan x \Rightarrow du = \sec^2 x dx \\ x=0 \Rightarrow u = \tan 0 = 0 \\ x=\pi/4 \Rightarrow u = \tan \pi/4 = 1 \end{array} \right\} \int_0^1 \sqrt{u} du = \frac{2}{3} u^{3/2} \Big|_0^1 = \frac{2}{3} [1^{3/2} - 0^{3/2}] = \frac{2}{3}$$

$$11. \int_{\pi^2}^{4\pi^2} \frac{3 \sin \sqrt{x}}{\sqrt{x}} dx \quad \left. \begin{array}{l} u = \sqrt{x} \Rightarrow 2 du = \frac{1}{\sqrt{x}} dx \\ x=\pi^2 \Rightarrow u = \pi, x=4\pi^2 \Rightarrow u = 2\pi \end{array} \right\} 2 \cdot 3 \int_{\pi}^{2\pi} \sin u du = -6 \cos u \Big|_{\pi}^{2\pi} = -6 [1 - (-1)] = 12$$

$$12. \int_{-1}^4 \frac{y dy}{\sqrt{5+y}} \quad \left. \begin{array}{l} u = 5+y \Rightarrow du = dy \text{ \& } y = u-5 \\ y=-1 \Rightarrow u=4, y=4 \Rightarrow u=9 \end{array} \right\} \int_4^9 \frac{u-5}{\sqrt{u}} du = \int_4^9 \sqrt{u} - \frac{5}{\sqrt{u}} du = \left[\frac{2}{3} u^{3/2} - 10 u^{1/2} \right]_4^9 = \left(\frac{2}{3} \cdot 27 - 10 \cdot 3 \right) - \left(\frac{16}{3} - 20 \right) = \frac{38}{3} - 10 = \frac{8}{3}$$

$$13. \int_{-2}^2 x \sqrt{16-x^4} dx \quad \left. \begin{array}{l} u = x^2 \Rightarrow \frac{1}{2} du = x dx \\ x=-2 \Rightarrow u=4 \\ x=2 \Rightarrow u=4 \end{array} \right\} \frac{1}{2} \int_4^4 \sqrt{16-u^2} du = 0$$

¹ Hint: Let $u = x^2$

14. $\int_{\pi/3}^{\pi/2} \sin \theta \sqrt{1-4\cos^2 \theta} d\theta$

$u = 2\cos \theta \Rightarrow du = -2\sin \theta d\theta$
 $x = \pi/3 \Rightarrow u = 2 \cdot \frac{1}{2} = 1$
 $x = \pi/2 \Rightarrow u = 2 \cdot 0 = 0$

$\left. \begin{aligned} & -\frac{1}{2} \int_1^0 \sqrt{1-u^2} du = -\frac{1}{2} \left(-\frac{\pi}{4}\right) = \pi/8 \\ & \uparrow \text{This is a semicircle:} \end{aligned} \right\}$



15. $\int_0^{\pi/6} \sin^4 \theta \cos^3 \theta d\theta = \int_0^{\pi/6} \sin^4 \theta \cos^2 \theta \cos \theta d\theta = \int \sin^4 \theta (1-\sin^2 \theta) \cos \theta d\theta$

$u = \sin \theta \Rightarrow du = \cos \theta d\theta$
 $\theta = 0 \Rightarrow u = 0$
 $\theta = \pi/6 \Rightarrow u = \sin \pi/6 = 1/2$

$\int_0^{1/2} u^4 (1-u^2) du = \int_0^{1/2} u^4 - u^6 du = \left[\frac{u^5}{5} - \frac{u^7}{7} \right]_0^{1/2} = \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7}$

16. $\int_0^3 |1-x^2| dx$

$= \int_0^1 (1-x^2) dx + \int_1^3 -(1-x^2) dx$
 $= \left[x - \frac{x^3}{3} \right]_0^1 + \left[-x + \frac{x^3}{3} \right]_1^3 = \left(1 - \frac{1}{3}\right) - 0 + \left(-3 + 9\right) - \left(-1 + \frac{1}{3}\right) = \frac{22}{3}$

17. $\int_0^{2\pi/3} |\cos x| dx$

$= \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{2\pi/3} -\cos x dx = \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{2\pi/3}$
 $= [1 - 0] + \left[-\frac{\sqrt{3}}{2} - (-1)\right] = 2 - \frac{\sqrt{3}}{2}$

18. $\int_{-2}^1 \sqrt{1+|x|} dx$

$= \int_{-2}^0 \sqrt{1-x} dx + \int_0^1 \sqrt{1+x} dx = \left[-\frac{2}{3}(1-x)^{3/2} \right]_{-2}^0 + \left[\frac{2}{3}(1+x)^{3/2} \right]_0^1$
 $= -\frac{2}{3} \left[1 - 3^{3/2} \right] + \frac{2}{3} \left[2^{3/2} - 1 \right]$
 $= \frac{2}{3} \left[-2 + 3^{3/2} + 2^{3/2} - 1 \right]$

19. Find the area between the curve $y = x^2 - x$ and the x-axis on the interval $[0, 2]$.

$A = -\int_0^1 (x^2 - x) dx + \int_1^2 (x^2 - x) dx$
 $= -\left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2$
 $= -\left[\left(\frac{1}{3} - \frac{1}{2}\right) - (0 - 0) \right] + \left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) \right]$
 $= -\frac{1}{6} + \frac{7}{3} - \frac{3}{2} = \frac{-1 + 14 - 9}{6} = \frac{4}{6} = \frac{2}{3}$

20. Evaluate the integral $\int_{-3}^1 \sqrt{3-2x-x^2} dx$ completing the square:

$3 - 2x - x^2 = 3 - (x^2 + 2x) = 3 - (x^2 + 2x + 1 - 1) = 3 + 1 - (x^2 + 2x + 1) = 4 - (x+1)^2$

$u = x+1 \Rightarrow du = dx$
 $x = -3 \Rightarrow u = -3+1 = -2$
 $x = 1 \Rightarrow u = 1+1 = 2$

$= \int_{-2}^2 \sqrt{4-u^2} du$

$= \frac{\pi r^2}{2} = \frac{\pi \cdot 2^2}{2} = 2\pi$

² Hint: use $\cos^2 \theta = 1 - \sin^2 \theta$