

Homework Set 23 : Solutions

- 1) Find L of $y = 3x^{3/2} - 1$ from $x=0$ to $x=1$.

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2} dx & f'(x) &= 3 \cdot \frac{3}{2} x^{1/2} = \frac{9}{2} x^{1/2} \Rightarrow [f'(x)]^2 = \frac{81}{4} x \\ &= \int_0^1 \sqrt{1 + \frac{81}{4} x} dx = \left. \frac{2}{3} \cdot \frac{4}{81} \left(1 + \frac{81}{4} x\right)^{3/2} \right|_0^1 = \frac{8}{243} \left[\left(\frac{85}{4}\right)^{3/2} - 1 \right] \\ &= \frac{8}{243} \left[\frac{85^{3/2}}{8} - 1 \right] = \frac{8}{243} \left[\frac{85^{3/2} - 8}{8} \right] = \frac{85\sqrt{85} - 8}{243} \end{aligned}$$

- 2) Find L for $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y=0$ to $y=1$

$$\begin{aligned} L &= \int_c^d \sqrt{1 + [g'(y)]^2} dy & g'(y) &= \frac{1}{3} \cdot \frac{3}{2} (y^2 + 2)^{1/2} \cdot 2y = y(y^2 + 2)^{1/2} \\ & & [g'(y)]^2 &= y^2(y^2 + 2) \\ & & 1 + [g'(y)]^2 &= y^2(y^2 + 2) + 1 = y^4 + 2y^2 + 1 = (y^2 + 1)^2 \\ &= \int_0^1 \sqrt{(y^2 + 1)^2} dy = \left. \frac{y^3}{3} + y \right|_0^1 = \left(\frac{1}{3} + 1\right) - 0 = \frac{4}{3} \end{aligned}$$

- 3) Find L for $y = x^{2/3}$ from $x=1$ to $x=8$.

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2} dx & f'(x) &= \frac{2}{3} x^{-1/3} & [f'(x)]^2 &= \frac{4}{9} x^{-2/3} \\ & & 1 + [f'(x)]^2 &= 1 + \frac{4}{9} x^{-2/3} = 1 + \frac{4}{9x^{2/3}} = \frac{9x^{2/3} + 4}{9x^{2/3}} \\ &= \int_1^8 \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}} dx = \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} dx & \text{let } u &= 9x^{2/3} + 4 \\ & & \text{so } \frac{du}{6} &= x^{-1/3} dx & 6 &\leftarrow \dots! \\ & & x=1 &\Rightarrow u = 9 \cdot 1 + 4 = 13 \\ & & x=8 &\Rightarrow u = 9 \cdot 8^{2/3} + 4 = 40 \\ &= \frac{1}{3} \cdot \frac{1}{6} \int_{13}^{40} \sqrt{u} du = \frac{1}{18} \cdot \frac{2}{3} u^{3/2} \Big|_{13}^{40} = \frac{1}{27} (40^{3/2} - 13^{3/2}) \end{aligned}$$

4) Find L for $y = \frac{x^6+8}{16x^2}$ from $x=2$ to $x=3$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad f(x) = \frac{x^4}{16} + \frac{1}{2x^2} \Rightarrow f'(x) = \frac{x^3}{4} - \frac{1}{x^3}$$

$$[f'(x)]^2 = \frac{x^6}{16} - 2 \cdot \frac{x^3}{4} \cdot \frac{1}{x^3} + \frac{1}{x^6} = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}$$

$$= \int_2^3 \sqrt{1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}} dx = \int_2^3 \sqrt{\frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6}} = \int_2^3 \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2} dx$$

$$= \int_2^3 \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx = \left[\frac{x^4}{16} - \frac{1}{2x^2}\right]_2^3 = \left(\frac{3^4}{16} - \frac{1}{18}\right) - \left(\frac{16}{16} - \frac{1}{8}\right) = \frac{595}{144}$$

5) Find L for $24xy = y^4 + 48$ from $y=2$ to $y=4$.

$$x = g(y) = \frac{y^4}{24y} + \frac{48}{24y} = \frac{y^3}{24} + \frac{2}{y} \Rightarrow g'(y) = \frac{y^2}{8} - \frac{2}{y^2}$$

$$1 + [g'(y)]^2 = 1 + \left[\frac{y^2}{8} - \frac{2}{y^2}\right]^2 = 1 + \frac{y^4}{64} - 2 \cdot \frac{y^2}{8} \cdot \frac{2}{y^2} + \frac{4}{y^4} = \frac{y^4}{64} + \frac{1}{2} + \frac{4}{y^4}$$

$$L = \int_2^4 \sqrt{\frac{y^4}{64} + \frac{1}{2} + \frac{4}{y^4}} dy = \int_2^4 \sqrt{\left(\frac{y^2}{8} + \frac{2}{y^2}\right)^2} dy = \left[\frac{y^3}{24} - \frac{2}{y}\right]_2^4$$

$$= \left(\frac{4^3}{24} - \frac{2}{4}\right) - \left(\frac{2^3}{24} - \frac{2}{2}\right) = \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 = \frac{17}{6}$$

6) $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y=1$ to $y=4$.

$$g'(y) = \frac{y^3}{2} - \frac{y^{-3}}{2} \Rightarrow 1 + [g'(y)]^2 = 1 + \left[\frac{y^6}{4} - \frac{1}{2} + \frac{y^{-6}}{4}\right] = \left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2$$

$$L = \int_1^4 \sqrt{\left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2} dy = \left[\frac{y^4}{8} - \frac{1}{4y^2}\right]_1^4 = \left(\frac{256}{8} - \frac{1}{64}\right) - \left(\frac{1}{8} - \frac{1}{4}\right)$$

$$= \frac{2055}{64}$$