

MATH 100
Midterm Examination
October 25, 2013

Instructions:

- ◆ You have 90 minutes to complete the examination.
- ◆ All questions are of equal value, but not of equal difficulty. Choose well, and use your time wisely!
- ◆ A box **must** be placed around your final answer to each question.
- ◆ Part marks can only be given if your rough work is shown. No marks will be given for unsupported answers.
- ◆ If you are unsure of anything, ask!

1) Find $\frac{d}{dx} [\sin^2 x + \cos^2 x] = \frac{d}{dx} [1] = 0$

2) Find $\lim_{x \rightarrow 0} \frac{x^{1/5}}{4x + \sin x} = \lim_{x \rightarrow 0} \frac{1}{\frac{4x}{x} + \frac{\sin x}{x}} = \frac{1}{4+1} = \frac{1}{5}$

3) Find $\frac{d}{dx} \left[\frac{x^3 - 2}{x \sec x} \right] = \frac{x \sec x \cdot 3x^2 - (x^3 - 2)[x \cdot \sec x \tan x + \sec x \cdot 1]}{(x \sec x)^2}$

4) Given that $y = 2x^2 - 1$ find a) the average rate of change of y with respect to x over the interval $[1, 4]$, and b) the instantaneous rate of change of y when $x = 4$.

a) $\frac{\Delta y}{\Delta x} = \frac{(2 \cdot 4^2 - 1) - (2 \cdot 1^2 - 1)}{4 - 1} = \frac{31 - 1}{3} = 10$ b) $f'(4) = 4x|_{x=4} = 16$

5) Find $\lim_{x \rightarrow +\infty} \frac{x^2 - x}{(x+1)^2} = \lim_{x \rightarrow +\infty} \frac{x^2}{x^2} = \lim_{x \rightarrow +\infty} 1 = 1$

6) Locate and classify the discontinuities, if any, in $f(x) = \frac{x+1}{x^2+x} = \frac{x+1}{x(x+1)}$ hole @ $x = -1$
asymptote @ $x = 0$

7) Find $\lim_{y \rightarrow 4} \frac{4-y}{2-\sqrt{y}} = \lim_{y \rightarrow 4} \frac{2+\sqrt{y} \cdot 2-\sqrt{y}}{2-\sqrt{y}} = 2+\sqrt{4} = 4$

8) Find $\lim_{x \rightarrow -1} \frac{x(x-1)}{x^2+2x+1} \rightarrow \frac{2}{0} \quad \lim_{x \rightarrow -1^+} \frac{x(x-1)}{(x+1)^2} \rightarrow \frac{2}{0^+} \quad \lim_{x \rightarrow -1^-} \frac{x(x-1)}{(x+1)^2} \rightarrow \frac{2}{0^-} \therefore +\infty$

9) Find $f'(x)$ if $f(x) = \frac{1}{2} \cos^4(x^5 - \sqrt{3}x)$ $f'(x) = \frac{1}{2} \cdot 4 \cos^3(x^5 - \sqrt{3}x) \cdot -\sin(x^5 - \sqrt{3}x) \cdot (5x^4 - \frac{1}{2\sqrt{3}x} \cdot 3)$

10) Use Newton's method to approximate $\sqrt[3]{26} \Rightarrow f(x) = x^3 - 26 \quad f'(x) = 3x^2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_3 = 2.962496068$

$x_0 = 3 \quad (\because 3^3 = 27)$

$x_4 = "$

$x_1 = 3 - \frac{3^3 - 26}{3 \cdot 3^2} = 2.962962963$

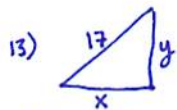
$x_2 = 2.962496142$

$$11) y = \frac{x(x-1)}{(x+1)^2} \quad y' = \frac{(x+1)^2(x-1 + (x-1) \cdot 1) - x(x-1) \cdot 2(x+1)}{(x+1)^4} \quad y'' = \frac{(x+1)^3 - (3x-1)3(x+1)^2}{(x+1)^4}$$

$$= \frac{2x^2 - x + 2x - 1 - 3x^2 + 3x}{(x+1)^3} = \frac{-x^2 + 4x - 1}{(x+1)^3}$$

$$= \frac{3x-1}{(x+1)^3}$$

$$= \frac{3x + \frac{1}{3} - \frac{1}{3} - \frac{1}{3}}{(x+1)^3} = \frac{3x - \frac{2}{3}}{(x+1)^3} = \frac{9x^2 - 2}{3(x+1)^3}$$



$$\frac{dx}{dt} = +5$$

$$\frac{dy}{dt} \Big|_{y=6} = ?$$

$$x^2 + y^2 = 17^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$= -\frac{\sqrt{17^2 - 6^2}}{6} \cdot 5$$

$$\approx -13.25/s$$

$$= -\frac{5\sqrt{253}}{6}$$

11) Find $\frac{d^2y}{dx^2}$ if $y = \frac{x(x-1)}{(x+1)(x+1)}$. (Hint: You will want to simplify the result for later use.)

$$- \sin(xy^2) [x \cdot 2y y' + y^2 \cdot 1] = y'$$

$$\frac{dy}{dx} = \frac{y^2 \sin(xy^2)}{1 + \sin(xy^2) \cdot 2xy}$$

12) Find $\frac{dy}{dx}$ if $\cos(xy^2) = y$.

13) A 17-foot ladder is leaning against a wall. If the bottom of the ladder is pulled away from the wall at a rate of 5 ft/sec, how fast will the top of the ladder be sliding down the wall when it is 6 feet above the ground?

14) Evaluate: $\lim_{x \rightarrow 2} \frac{\sin(2x-4)}{x^2-4} = \lim_{x \rightarrow 2} \frac{\sin(2(x-2))}{2(x-2)(x+2)} = \frac{2}{2+2} = \frac{1}{2}$

15) Sketch $f(x) = \frac{x^2-x}{(x+1)^2}$. Label all points of interest. (Hint: You may find your results from problems 5, 8 and 11 helpful.)

No symmetry.
y-int: 0
x-int: $x(x-1)=0$
H.A.: $x=-1$
V.A.: $x=1, \rightarrow +\infty$

fr. ii) stationary point $3x-1=0 \Rightarrow x=1/3$
inflection: $x=1, y=0$

16) Use the definition of the derivative to prove either that $\frac{d}{dx} [\sin x] = \cos x$ or that

$$\frac{d}{dx} \left[\frac{1}{\sqrt{x}} \right] = -\frac{1}{2\sqrt{x^3}}$$

17) $\max S = kwh^2$
 $w^2 + h^2 = D^2 - w^2$
 $\therefore S = kw(D^2 - w^2)$
 $= k w D^2 - k w^3$
 $\frac{dS}{dw} = k D^2 - 3k w^2 = 0$

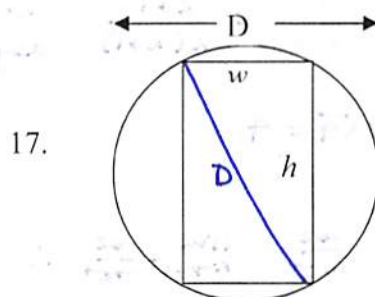
$$w = \frac{D}{\sqrt{3}}$$

$$h = \sqrt{D^2 - \left(\frac{D}{\sqrt{3}}\right)^2}$$

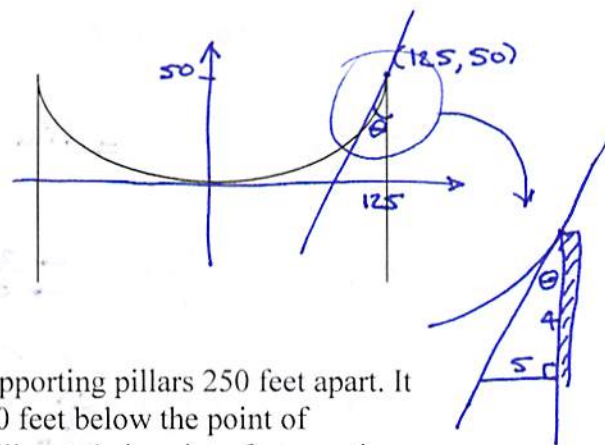
$$= \sqrt{\frac{2}{3}} D$$

global max

17) Alex and Kim have felled some trees and now want to mill them up into rectangular beams by trimming off the rounded sides. The question they have is how much of each log to trim off to get the strongest beams. An engineer tells them that the strength of a beam with a rectangular cross section varies directly with its width and as the square of its height. What are the dimensions (i.e. width, w , and height, h) of the strongest beam that can be sawn out of a round log that is D cm across?



18.



18) The main cable of a suspension bridge is attached to supporting pillars 250 feet apart. It hangs in the form of a parabola with the lowest point 50 feet below the point of suspension. Find the angle between the cable and the pillar at their point of connection.

parabola $\Rightarrow y = kx^2$

$$50 = k \cdot 125^2$$

$$250 = k \cdot (5 \cdot 25)^2$$

$$k = \frac{2}{25^2}$$

$$m_{\tan} = 2kx = 2 \cdot \frac{2}{25^2} \cdot 125 = \frac{20}{25} = \frac{4}{5} = \frac{\text{rise}}{\text{run}}$$

$$\therefore \tan \theta = 5/4$$

$$\theta = \tan^{-1} 5/4$$

$$\approx 51.34^\circ$$