

Exercise set 11.1: Combinatorics

1. Evaluate each expression by hand (check by calculator if you like):

a. $\frac{12!}{11!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 12$

b. $\frac{16!}{11!} = 16 \times 15 \times 14 \times 13 \times 12 = 524,160$

c. $\frac{10!}{5!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \overset{2}{\cancel{6 \cdot 5}} \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}} = 1,260$

d. $\frac{4!110!}{105!2!} = \frac{4 \cdot 3 \cdot \cancel{2 \cdot 1} \cdot 110 \cdot 109 \cdot 108 \cdot 107 \cdot 106}{\cancel{2 \cdot 1}} = 1.762437917 \times 10^6$

2. Write each expression in terms of factorials:

a. $24 \times 23 \times 22 \times 21 \times 20 = \frac{24!}{19!}$

b. $\frac{1}{10 \times 9 \times 8} = \frac{7!}{10!}$

3. A restaurant has a menu featuring 4 appetizers, 5 entrees, and 2 desserts. How many different three-course meals can be ordered?

$$4 \times 5 \times 2 = 40$$

4. Along a migration route there are three trails from A to B, 2 trails from B to C, and 4 trails from C to D. How many possible routes are there for an individual animal to travel from A to D?

$A \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} B \begin{array}{c} \text{---} \\ \text{---} \end{array} C \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} D \quad 3 \times 2 \times 4 = 24$

5. A classroom has 22 students, 12 boys and 10 girls.

- a. How many ways can the teacher select 2 students to go help a guest speaker carry in equipment?

$${}_{22}C_2 = 231$$

- b. In how many of those ways are both students boys?

$${}_{12}C_2 = 66$$

- c. If the students are selected randomly what is the probability that both are boys?

$$P = \frac{n}{N} = \frac{66}{231} \approx 0.286$$

6. In how many ways can the nine members of a tribunal reach:

- a. A 6-3 decision?

$${}_9C_6 = 84$$

- b. A majority decision? i.e. 6-3, 5-4, 7-2, 8-1 or 9-0

$${}_9C_6 + {}_9C_5 + {}_9C_7 + {}_9C_8 + {}_9C_9 = 256$$

Useful facts about a deck of cards: 4 suits (clubs, diamonds, hearts and spades) each in 13 denominations (Ace, 2, 3, ..., Queen, King) for a total of 52 cards.

7. How many possible five-card poker hands can be dealt from a full deck? $52C_5 = 2,598,960 = \sim$
8. How many ways can each type of poker hand be dealt from a full deck? What is the corresponding probability of each hand?

Poker Hand Rankings

1. Royal Flush

A, K, Q, J, 10
all of the
same suit



2. Straight Flush

Any five card
sequence in
the same suit



3. Four of a Kind

All four cards of
the same rank



4. Full House

Three of a kind
combined with
a pair



5. Flush

Any five cards of
the same suit,
but not in
sequence



6. Straight

Five cards in
sequence, but
not in the same
suit



7. Three of a Kind

Three cards of
the same rank



8. Two Pair

Two separate
pairs



9. Pair

Two cards of
the same rank



10. High Card

Otherwise
unrelated cards
ranked by the
highest single card



1. # of possible 5-card hands = ${}_{52}C_5 = 2,598,960$ (=N for convenience below)

2. Royal Flush: Only 4 possible Royal Flushes (one in each suit)
 $\therefore p = 4/N \approx 0.00000153908$

Straight Flush: 10 possibilities in each suit; ($\because$ Ace can be low or high)

A-2-3-4-5
2-3-4-5-6
3-4-5-6-7
4-5-6-7-8
5-6-7-8-9
6-7-8-9-10
7-8-9-10-J
8-9-10-J-Q
9-10-J-Q-K
10-J-Q-K-A

$\therefore$ 10 possibilities $\times$ 4 suits $\Rightarrow p = 4 \cdot 10/N \approx 0.0000153908$

Four of a kind 13 possibilities for the rank of the four
AAAA B $\times$ 48 possibilities for the other card ($\because 52-4$)
 $\Rightarrow p = 13 \cdot 48/N \approx 0.000240096$

Full House ${}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_1 \cdot {}_4C_2 \Rightarrow 13 \cdot 4 \cdot 12 \cdot 6 = 3744$
AAAB B

choose 1st rank $\uparrow$
choose 3 cards of 1st rank $\uparrow$
choose 2nd rank $\uparrow$
choose 2 cards of that rank $\uparrow$

$\therefore p = 3744/N$
 ≈ 0.00144058

Flush

${}_4C_1 \cdot {}_{13}C_5 = 4 \cdot 1287 = 5148$
choose the suit $\uparrow$
choose 5 cards from that suit $\uparrow$

$\therefore p = 5148/N$
 ≈ 0.0019808

N.B. To exclude straight flushes $p = (5148 - 40)/N$
 ≈ 0.001965402

Straight

10 patterns (see straight flush) $\times$ 4 possible cards at each position in the pattern (e.g. 4 possible 5s)

$$\therefore p = \frac{10 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{N} = \frac{10240}{N} \approx 0.00394004$$

N.B. To exclude straight flushes: $p = \frac{10240 - 40}{N} \approx 0.00392465$

Three of a kind AAABC: ${}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_2 \cdot {}_4C_1 \cdot {}_4C_1 = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4$
 $= 54,912$

choose rank of triple

choose 3 cards of that rank

choose 2 other ranks

choose 1 card of 1st other rank

choose 1 card of 2nd other rank

$$\therefore p = 54,912/N \approx 0.0211285$$

Two pairs AABBC: ${}_{13}C_2 \cdot {}_4C_2 \cdot {}_4C_2 \cdot {}_{12}C_1 \cdot {}_4C_1 = 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4$
 $= 123,552$

choose 2 ranks

choose 2 cards of 1st rank

choose 2 cards of 2nd rank

choose a 3rd rank

choose 1 card of 3rd rank

$$\therefore p = 123,552/N \approx 0.04753902$$

Pair AABCD:

$${}_{13}C_1 \cdot {}_4C_2 \cdot {}_{12}C_3 \cdot {}_4C_1 \cdot {}_4C_1 \cdot {}_4C_1 = 13 \cdot 6 \cdot 220 \cdot 4 \cdot 4 \cdot 4$$

 $= 1098,240$

choose rank of pair

choose 2 cards of pair rank

choose 3 other ranks

choose 1 card of each other rank

$$\therefore p = 0.422569027$$

High Card

choose 5 different ranks
but not a straight

$$\begin{array}{r} {}_{13}C_5 \\ - 10 \\ \hline 1277 \end{array}$$

choose any pattern of suits,
but not all the same

$$\begin{array}{r} 1277 \\ \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \\ - 4 \\ \hline 1020 \end{array}$$

$$\therefore p \approx 0.5011774$$