

Matrix Multiplication

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⊙ Scalars, vectors and matrices

Scalars are numbers, individual numeric values, e.g. 7, -4.9, 0.00382

Vectors are lists of numbers, and are referred to as row vectors or column vectors depending on whether they are oriented horizontally or vertically.

A five-element row vector: $[1.6 \quad 2.3 \quad 1.4 \quad 1.8 \quad 0.9]$

A three-element column vector: $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$

Matrices are tables or arrays of numbers. An $m \times n$ matrix is one with m rows and n columns.

A 2×2 matrix: $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

A 3×5 matrix: $\begin{bmatrix} 2 & 4 & -3 & 5 & 6 \\ 1 & 1 & 7 & 2 & 4 \\ 8 & -3 & -2 & 1 & 6 \end{bmatrix}$

A 1×3 matrix: $[0 \quad 1 \quad 0]$.

As you can see a vector is just a 1-dimensional matrix, but they are so common that it's convenient to have a distinct name for them.

⊙ Adding and subtracting matrices

We add or subtract matrices by adding or subtracting the corresponding entries in them, e.g.

$$\begin{bmatrix} 10 & 15 & 20 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 - 1 & 15 - 2 & 20 - 3 \end{bmatrix} = \begin{bmatrix} 9 & 13 & 17 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 15 \\ 20 & 25 & 30 \end{bmatrix} = \begin{bmatrix} 1 + 5 & 2 + 10 & 3 + 15 \\ 4 + 20 & 5 + 25 & 6 + 30 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 24 & 30 & 36 \end{bmatrix}$$

Notice that we can only add or subtract matrices of the same size since they have to have corresponding entries.

⊙ Multiplying matrices (false start)

Multiplying matrices is not as straightforward as adding or subtracting them. It's a somewhat lengthy process, but it can be reduced to a series of vector multiplications, so let's see how to do those first.

⊙ Multiplying vectors

We multiply vectors by finding *the sum of the products of their entries*. It's easier to see than to read so look at these examples:

$$\begin{bmatrix} 4 \end{bmatrix} \times \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 5 \\ 10 \end{bmatrix} = [(1 \times 5) + (2 \times 10)] = [5 + 20] = [25]$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = [(1 \times 5) + (2 \times 10) + (3 \times 15)] = [5 + 20 + 45] = [70]$$

Here's how to see that this is indeed the sum of the products of the entries:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = [(1 \times 5) + (2 \times 10) + (3 \times 15)] = [5 + 20 + 45] = [70]$$

products of entries sum of products result

To make sure we get the entries paired correctly we often write the multiplication out like this on paper with the second matrix raised in the air so we can fit the result underneath it and draw arrows to keep our pairings straight.

$$\begin{array}{ccc} & & \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} \\ & \nearrow & \\ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} & & \begin{bmatrix} 70 \end{bmatrix} \end{array}$$

The dashed arrows are a helpful visual aid to make sure we pair up the correct elements from the two vectors.

Now that we know how to multiply vectors we are ready to tackle matrix multiplication.

⊙ Multiplying matrices

We multiply matrices by performing a series of vector multiplications. Specifically, each entry in the result matrix is the product of **the vector to the left** of it with **the vector above** it. Here, take a look:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = \begin{bmatrix} 35 & 50 \\ 75 & 110 \end{bmatrix}$$

Here are the vector multiplications that produced each entry in that result:

$$\left[\begin{array}{cc|c} \begin{array}{cc} \begin{bmatrix} 1 & 2 \end{bmatrix} & \begin{bmatrix} 5 \\ 15 \end{bmatrix} & \mathbf{35} \\ \begin{bmatrix} 1 & 2 \end{bmatrix} & \begin{bmatrix} 10 \\ 20 \end{bmatrix} & \mathbf{50} \end{array} & \begin{array}{cc} \begin{bmatrix} 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 \\ 15 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} & \begin{bmatrix} 10 \\ 20 \end{bmatrix} & \mathbf{110} \end{array} \right]$$

If that's a little overwhelming, let's look at finding just one value in a result matrix. Consider this matrix multiplication,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix}$$

If we write the product out we can see the result will be a 2x2 matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's just find element b in the upper right cell of the result.

It will be the product of **the vector to the left of it** with **the vector above it**:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This vector product is $(1 \times 10) + (2 \times 20) + (3 \times 30) = 140$ so the value of b is 140.

We do the same thing for every element in the result matrix: **look to the left for a row**, and **look above for a column**, and multiply them to get the result value (by finding the sum of their pairwise products). So,

$$a = (1 \times 5) + (2 \times 15) + (3 \times 25) = 110$$

$$c = (4 \times 5) + (5 \times 15) + (6 \times 25) = 245$$

$$d = (4 \times 10) + (5 \times 20) + (6 \times 30) = 320$$

And the complete result is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 10 \\ 15 & 20 \\ 25 & 30 \end{bmatrix} = \begin{bmatrix} 110 & 140 \\ 245 & 320 \end{bmatrix}$$

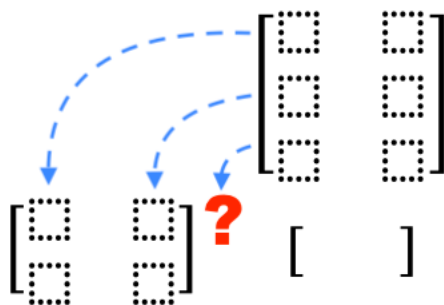
© The size of the result matrix

You might have noticed in the multiplication above that we multiplied a 2×3 matrix by a 3×2 matrix and got a 2×2 result. This is an instance of a general pattern: when you multiply an $m \times n$ matrix by an $h \times j$ matrix the result will be an $m \times j$ matrix, with m rows and j columns, i.e.

$$[m \times n] \times [h \times j] \rightarrow [m \times j]$$

⊙ Matrices that can't be multiplied

We've seen in the example above that matrices don't have to have the same size to be multiplied. Instead, because of the way we need to pair up entries from rows of the first matrix with entries in the columns of the second matrix, the number of columns in the first matrix has to be equal to the number of rows in the second matrix, so in the equation above n must be equal to h . What happens if they aren't? Let's see:



Oops, there's no column in the matrix on the left, for the entries in the bottom row of the matrix on the right to pair with. So if,

$$\text{Columns on the left} \neq \text{Rows on the right}$$

you can't multiply the matrices!

⊙ Just for completeness: One last thing about multiplying vectors

Way up above we multiplied a row vector by a column vector, and got a scalar result. We looked at that because it's the situation that arises when multiplying matrices, and that's what we were working toward. But, you *can* multiply a column vector by a row vector. Let's see what happens:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times [5 \quad 10 \quad 15] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 15 \\ 1 \times 5 & 1 \times 10 & 1 \times 15 \\ 2 \times 5 & 2 \times 10 & 2 \times 15 \\ 3 \times 5 & 3 \times 10 & 3 \times 15 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{bmatrix}$$

So the result of multiplying a column vector by a row vector is a matrix! (Not a scalar.)