

Exercise set 7.3

1. **The basics** Consider the function $f(x) = 1 + x^2 + x^3 + x^4 + x^5 + x^6$.

a. What is its rate of change when x is 0?

"rate of change" $\Rightarrow f'(x)$

$$f'(x) = 0 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + 6x^5$$

$$f'(0) = 0 + 0 + 0 + 0 + 0 = 0$$

b. What is its rate of change when x is 1?

$$f'(1) = 2 \cdot 1 + 3 \cdot 1^2 + 4 \cdot 1^3 + 5 \cdot 1^4 + 6 \cdot 1^5 = 20$$

c. What is its second derivative?

$$\begin{aligned} f''(x) &= 2 + 2 \cdot 3x^1 + 4 \cdot 3x^2 + 5 \cdot 4x^3 + 6 \cdot 5x^4 \\ &= 2 + 6x + 12x^2 + 20x^3 + 30x^4 \end{aligned}$$

d. What is its fifth derivative?

$$\begin{aligned} f''(x) &= 2 + 2 \cdot 3x^1 + 4 \cdot 3x^2 + 5 \cdot 4x^3 + 6 \cdot 5x^4 \\ &= 2 + 6x + 12x^2 + 20x^3 + 30x^4 \end{aligned}$$

$$\begin{aligned} f'''(x) &= 0 + 6 + 12 \cdot 2x + 20 \cdot 3x^2 + 30 \cdot 4x^3 \\ &= 6 + 24x + 60x^2 + 120x^3 \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= 0 + 24 + 60 \cdot 2x + 120 \cdot 3x^2 \\ &= 24 + 120x + 360x^2 \end{aligned}$$

$$\begin{aligned} f^{(5)}(x) &= 0 + 120 + 360 \cdot 2x \\ &= 120 + 720x \end{aligned}$$

2. **Higher order derivatives** Find $f'''(x)$ if $f(x) = 6x^5 - 4x^2$.

$$f(x) = 6x^5 - 4x^2$$

$$f'(x) = 6 \cdot 5x^4 - 4 \cdot 2x = 30x^4 - 8x$$

$$f''(x) = 30 \cdot 4x^3 - 8 = 120x^3 - 8$$

$$f'''(x) = 120 \cdot 3x^2 - 0 = 360x^2$$

3. **Higher order derivatives** Find $f'''(x)$ if $f(x) = \frac{1}{x}$

$$f(x) = 1/x = x^{-1}$$

$$f'(x) = -1x^{-1-1} = -x^{-2} = -1/x^2$$

$$f''(x) = -1 \cdot -2x^{-2-1} = 2x^{-3} = 2/x^3$$

$$f'''(x) = 2 \cdot -3x^{-3-1} = -6x^{-4}$$

4. **Graphing** Where does the graph of $y = \frac{x^2+9}{x}$ have a horizontal tangent line?

"horizontal tangent line"

$$\Rightarrow \text{slope} = 0$$

$$\Rightarrow m_{\text{tan}} = 0$$

$$\Rightarrow y' = 0$$

$$\begin{aligned} y = \frac{x^2+9}{x} &\Rightarrow y' = \frac{x(2x+0) - (x^2+9) \cdot 1}{x^2} \quad \text{by quotient rule} \\ &= \frac{2x^2 - x^2 - 9}{x^2} \\ &= \frac{x^2 - 9}{x^2} \end{aligned}$$

Now find value of x that makes this 0:

$$\cancel{x^2} \frac{x^2 - 9}{\cancel{x^2}} \stackrel{!}{=} 0 \cdot \cancel{x^2}$$

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

5. **HIV density** The viral load for a particular HIV patient undergoing treatment using ABT-538 is modeled by the function,

$$V = 96.39785(0.818656)^t$$

Where V is the viral load measured in RNA copies per mL and t is the time in days since treatment began. Find $V'(1)$ and $V'(10)$ and interpret them.

Given: $V = 96.39785(0.818656)^t$

Find & interpret: $V'(1)$ & $V'(10)$

Sol'n: $V'(t) = 96.39785(0.818656)^t \cdot \ln 0.818656$

$$\begin{aligned} V'(1) &= 96.39785(0.818656)^1 \cdot \ln 0.818656 \\ &= -15.79054136 \end{aligned}$$

$$\begin{aligned} V'(10) &= 96.39785(0.818656)^{10} \cdot \ln 0.818656 \\ &= -2.608014863 \end{aligned}$$

On day 1 the viral load is going ^{down} relatively quickly at
-15.79054136 RNA copies/mL/day.

On day 10 it is still decreasing but at the slower rate of
-2.608014863 RNA copies/mL/day.

6. **Ice and water** Why does ice float on water? Because the density of ice is lower! In fact, water is the only common liquid whose maximal density occurs above its freezing temperature, and this has important biological implications, e.g. stratification of water in lakes by temperature, and preventing ice from forming at the bottoms of lakes and them freezing solid. Between 0°C and 30°C the volume V (in cubic centimetres) of 1 kg of water at a temperature T is given by the formula,

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.¹

Given: $V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$

Find: T to maximize density $\Rightarrow T$ to minimize V

Sol'n: Check ① stationary points, ② other critical points, ③ interval end points.

$$\textcircled{1} \frac{dV}{dT} = 0 - \underbrace{0.06426}_c + \underbrace{0.0085043 \cdot 2T}_b - \underbrace{0.0000679 \cdot 3T^2}_a \stackrel{!}{=} 0$$

Solutions at $T = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-2(0.0085043) \pm \sqrt{(2(0.0085043))^2 - 4(-0.0000679 \cdot 3)}}{2(-0.0000679 \cdot 3)}$$

$$= \boxed{3.966514624} \text{ \& } 79.53176716$$

$V = \boxed{937.3427923}$ $\uparrow$ outside range $0-30$

② No other critical points eg discontinuities, cusps, vertical tangent lines.

③ @ $x=0$ $V = \boxed{999.87}$
 @ $x=30$ $V = \boxed{1003.76277}$

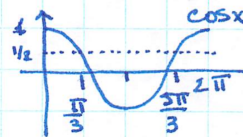
$\therefore T$ to maximize density (i.e. minimize volume) is $\approx 3.966514624^{\circ}\text{C}$

¹ Hint: Maximizing the density is equivalent to minimizing the volume.

7. **EVT** Find the maximum and minimum values of $f(x) = x - 2\sin x$ on the interval $[0, 2\pi]$.

Find: Min and max values of $f(x) = x - 2\sin x$ on $[0, 2\pi]$

Sol'n: $f'(x) = 1 - 2\cos x \stackrel{!}{=} 0 \Rightarrow \cos x = 1/2 \Rightarrow x = \cos^{-1}(1/2) = \pi/3 \text{ \& } 5\pi/3$



@ $x = \frac{\pi}{3}$ $f(\pi/3) = \cancel{\pi/3} - 2\sin \frac{\pi}{3} \approx -0.684853256$

@ $x = \frac{5\pi}{3}$ $f(5\pi/3) = \frac{5\pi}{3} - 2\sin \frac{5\pi}{3} \approx 6.968038564$

End points:

@ $x = 0$ $f(0) = 0 - 2\sin 0 = 0 - 2 \cdot 0 = 0$

@ $x = 2\pi$ $f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi \approx 6.283185307$

$\therefore f_{\max}$ occurs at $x = \frac{5\pi}{3}$ and is ≈ 6.968038564

$f_{\min}$ occurs at $x = \frac{\pi}{3}$ and is ≈ -0.684853256