

Exercise set 8.1 Solutions: Applications of the derivative

1. **Photosynthesis** The rate (in mg of carbon/m³/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

Given: $P = \frac{100I}{I^2 + I + 4}$

Find: I to maximize P

Soln: ① $\frac{dP}{dI} = \frac{(I^2 + I + 4) \cdot 100 - 100I(2I + 1)}{(I^2 + I + 4)^2} \stackrel{!}{=} 0$

$$100(I^2 + I + 4) - 100I(2I + 1) = 0$$

$$I^2 + I + 4 - 2I^2 - I = 0$$

$$-I^2 + 4 = 0$$

$$-I^2 = -4$$

$$I = \pm 2 \quad (\text{but we ignore } -2)$$

$$P(2) = \frac{100 \cdot 2}{(2)^2 + 2 + 4}$$

$$= \frac{100 \cdot 2}{4 + 2 + 4} = 20$$

② No other ~~critical~~ critical points.

③ Endpoints: $I=0 \Rightarrow P = \frac{100 \cdot 0}{0^2 + 0 + 4} = \frac{0}{4} = 0$

$$I \rightarrow \infty \Rightarrow P \rightarrow 0$$

$\therefore P_{\max}$ occurs at $I = 2$ where $P = 20$ (mg of C/m³/h)

The rate of photosynthesis is maximized for a light intensity of 2 thousand foot-candles.

2. **Crop yield** A model used for the yield Y of an agricultural crops as a function of the nitrogen level N in the soil is

$$Y = \frac{27N}{1+N^2}$$

What nitrogen level gives the best yield?

Given: $Y = \frac{27N}{1+N^2}$

Find: N to maximize Y

Sol'n: $\frac{dY}{dN} = \frac{(1+N^2)27 - 27N \cdot 2N}{(1+N^2)^2} \stackrel{!}{=} 0$

$$27(1+N^2) - 27 \cdot 2N^2 = 0$$

$$1 - N^2 = 0$$

$$N^2 = 1 \Rightarrow N = \pm \sqrt{1} = \pm 1$$

$$P(1) = \frac{27 \cdot 1}{1+1^2} = \frac{27}{2}$$

No other critical points.

Endpoints: $N=0 \Rightarrow Y = \frac{27 \cdot 0}{1+0^2} = \frac{0}{1} = 0$

$N \rightarrow \infty \Rightarrow Y = 0$

$\therefore$ max occurs @ $N=1$ and is $Y = 27/2$ (units not given)

3. **Disease virulence** The Kermack-McKendrick model for infectious disease transmission can be used to predict the population size P as a function of the disease's virulence. Virulence measures the deadliness of a disease. The population size P is high when virulence v is low because it is not very deadly. Surprisingly P is also high when v is high because then the disease is so deadly it kills people before they can pass it on, so fewer people actually get infected. Suppose,

$$P(v) = \frac{10 + v + v^2}{1 + v}$$

for values of v from 0 to 10 inclusive. What virulence values produce the largest and smallest population sizes? What are those populations?

Given: $P(v) = \frac{10+v+v^2}{1+v}$, $0 \leq v \leq 10$

Find: v to max and min P

Sol'n: $P'(v) = \frac{(1+v) \cdot (1+2v) - (10+v+v^2)(1)}{(1+v)^2} \leq 0$

$$1+2v+2v^2-10-v-v^2=0$$

$$-9+2v+v^2=0$$

$$v = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot -9}}{2 \cdot 1} = \frac{-2 \pm \sqrt{40}}{2} = \frac{-2 \pm 2\sqrt{10}}{2} = -1 \pm \sqrt{10}$$

Ignore $-1-\sqrt{10}$ $\because$ it is < 0 and $\therefore$ outside the domain
 $P(-1+\sqrt{10}) \approx 5.32455532$

Endpoints: $P(0) = \frac{10+0+0^2}{1+0} = \frac{10}{1} = 10$

$$P(10) = \frac{10+10+10^2}{1+10} = \frac{120}{11} = 10.\bar{90}$$

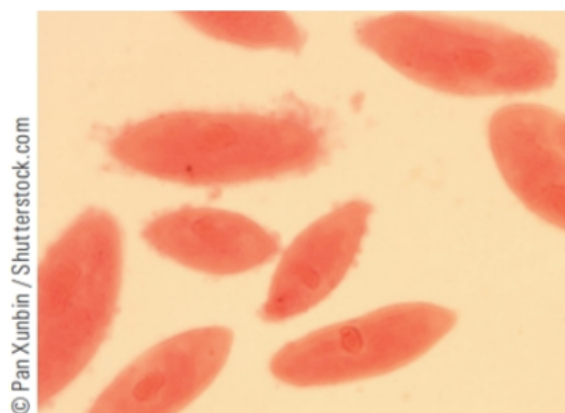
$\therefore P_{\max}$ occurs for virulence $v=10$

$P_{\min}$ occurs for virulence $v = -1+\sqrt{10} \approx 2.16227766$

4. **Gause's logistic model** In an experiment with the protozoan *Paramecium* the biologist G.F. Gause modeled the protozoan population size with the logistic function

$$P(t) = \frac{61}{1 + 31e^{-0.7944t}}$$

Where t is measured in days. According to this model, how fast was the population growing after 8 days?



Given: $P(t) = \frac{61}{1 + 31e^{-0.7944t}}$

Find: $P'(8)$

Solution: $P'(t) = \frac{(1 + 31e^{-0.7944t})^0 \cdot 0 - 61(0 + 31e^{-0.7944t}) \cdot -0.7944}{(1 + 31e^{-0.7944t})^2}$
 $= \frac{-61(31)(-0.7944)e^{-0.7944t}}{(1 + 31e^{-0.7944t})^2}$

$\therefore P'(8) = +2.350335048$ i.e. "on" day 8 the population is increasing by a bit more than 2 protozoa per day.