

Exercise set 9.3: Differential Equations

Solving differential equations can be difficult, but testing proposed solutions is easier: we can simply substitute the proposed solution into the differential equation and see if it satisfies it. For example to see if $y = \cos t$ is a¹ solution to $y'' + y = 0$ we calculate y'' ,

$$y = \cos t \Rightarrow y' = -\sin t \Rightarrow y'' = -\cos t$$

and substitute,

$$y'' + y = 0 \Rightarrow -\cos t + \cos t = 0$$

Since this equation is true, $y = \cos t$ is in fact a solution to the differential equation $y'' + y = 0$.

Test the following proposed solutions to see which ones satisfy the associated equations.

1. Equation: $y' - 2 = 0$. Proposed solution: $y = 2x$. $\Rightarrow y' = 2$

Substitute: $2 - 2 \stackrel{?}{=} 0$ $\checkmark$ $\therefore y = 2x$ is a solution.

2. Equation: $y' - 2 = 0$. Proposed solution: $y = 2x + 1$. $\Rightarrow y' = 2 + 0 = 2$

Sub: $2 - 2 \stackrel{?}{=} 0$ $\checkmark$ $\therefore y = 2x + 1$ is a solution.

3. Equation: $y' - 2 = 0$. Proposed solution: $y = 2x + p$. $\Rightarrow y' = 2 + 0 = 2$

Sub: $2 - 2 \stackrel{?}{=} 0$ $\checkmark$ $\therefore y = 2x + p$ is a general solution.

(What conclusion can you draw from the first three problems?)

That adding/changing a constant term does not affect solution status.

That DEs have multiple solutions.

4. Equation: $y' - y = 0$. Proposed solution: $y = x^2$. $\Rightarrow y' = 2x$

Sub: $2x - x^2 \stackrel{?}{=} 0$ Not in general $\therefore y = x^2$ is not a solution.

¹ It may not be the **only** solution.

5. Equation: $y' - y = 0$. Proposed solution: $y = e^x \Rightarrow y' = e^x$

sub $e^x - e^x \stackrel{?}{=} 0 \checkmark \therefore y = e^x$ is a solution.

6. Equation: $y'' + y = 0$. Proposed solution: $y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2$

sub: $2 + x^2 \stackrel{?}{=} 0$ Not in general so not a solution.

7. Equation: $y'' + y - 2 = 0$. Proposed solution: $y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2$

sub: $2 + x^2 - 2 \stackrel{?}{=} 0$
 $x^2 \stackrel{?}{=} 0$ Not in general.

chain rule

8. Equation: $y'' + 4y = 0$. Proposed solution: $y = \cos(4x) \Rightarrow y' = -\sin(4x) \cdot 4$
 $= -4 \sin(4x)$

sub: $-16 \cos 4x + 4 \cos 4x \stackrel{?}{=} 0$
 $-12 \cos 4x \stackrel{?}{=} 0$ No.

$\Rightarrow y'' = -4 \cdot \cos(4x) \cdot 4$
 $= -16 \cos 4x$

9. Equation: $y'' + y = 0$. Proposed solution: $y = \sin t + \cos t \Rightarrow y' = \cos t - \sin t$

sub: $\underbrace{-\cancel{\sin t} - \cancel{\cos t}}_0 + \underbrace{\cancel{\sin t} + \cancel{\cos t}}_0 \stackrel{?}{=} 0$
 $\stackrel{?}{=} 0$ Yes!

$\Rightarrow y'' = -\sin t - \cos t$

10. Equation: $y'' + y = 0$. Proposed solution: $y = 3 \sin t + 2 \cos t \Rightarrow y' = 3 \cos t - 2 \sin t$

sub: $\underbrace{-3 \cancel{\sin t} - 2 \cancel{\cos t}}_0 + \underbrace{3 \cancel{\sin t} + 2 \cancel{\cos t}}_0 \stackrel{?}{=} 0$

$\Rightarrow y'' = -3 \sin t - 2 \cos t$

$0 \stackrel{?}{=} 0 \checkmark$ Yes.

11. Equation: $y'' + 2y' + y = 0$. Proposed solution: $y = e^{-t}$. $\Rightarrow y' = e^{-t} \cdot -1 = -e^{-t}$ chain rule
 $\Rightarrow y'' = -e^{-t} \cdot -1 = e^{-t}$

Sub: $e^{-t} + 2 \cdot -e^{-t} + e^{-t} \stackrel{?}{=} 0$
 $2e^{-t} - 2e^{-t} \stackrel{?}{=} 0 \quad \checkmark \quad \text{Yes.}$

12. Equation: $y'' + 2y' + 2y = 0$. Proposed solution: $y = e^{-t} \sin t$. Product rule
 $\Rightarrow y' = e^{-t} \cdot \cos t + \sin t \cdot e^{-t} \cdot -1$
 $= e^{-t} \cos t - e^{-t} \sin t$

$\Rightarrow y'' = e^{-t} \cdot -\sin t + \cos t \cdot e^{-t} \cdot -1$
 $-(e^{-t} \cos t + \sin t \cdot e^{-t} \cdot -1)$
 $= -\cancel{e^{-t} \sin t} - e^{-t} \cos t - e^{-t} \cos t$
 $+ \cancel{e^{-t} \sin t}$
 $= -2e^{-t} \cos t$

Sub: $y'' + 2y' + 2y = 0$
 $-2e^{-t} \cos t + 2(e^{-t} \cos t - e^{-t} \sin t) + 2e^{-t} \sin t \stackrel{?}{=} 0$
 $-\cancel{2e^{-t} \cos t} + \cancel{2e^{-t} \cos t} - \cancel{2e^{-t} \sin t} + \cancel{2e^{-t} \sin t} \stackrel{?}{=} 0$
 $0 \stackrel{?}{=} 0 \quad \checkmark$
 Yes.