

Homework 7.3: Differential Equations

Solving differential equations can be difficult, but testing proposed solutions is easier: we can simply substitute the proposed solution into the differential equation and see if it satisfies it. For example to see if $y = \cos t$ is ^a solution to $y'' + y = 0$ we calculate y'' ,

$$y = \cos t \Rightarrow y' = -\sin t \Rightarrow y'' = -\cos t$$

and substitute,

$$y'' + y = 0 \Rightarrow -\cos t + \cos t = 0$$

Since this equation is true, $y = \cos t$ is in fact a solution to the differential equation $y'' + y = 0$.

Test the following suggested solutions to see which ones solve the associated equations.

1. Equation: $y' - 2 = 0$. Suggested solution: $y = 2x \Rightarrow y' = 2, 2 - 2 \stackrel{?}{=} 0 \checkmark \therefore$ solution
2. Equation: $y' - 2 = 0$. Suggested solution: $y = 2x + 1 \Rightarrow y' = 2, 2 - 2 \stackrel{?}{=} 0 \checkmark \therefore$ solution
3. Equation: $y' - 2 = 0$. Suggested solution: $y = 2x + \pi \Rightarrow y' = 2, 2 - 2 \stackrel{?}{=} 0 \checkmark \therefore$ solution

(What conclusion can you draw from the first three problems?) That adding/changing a constant term does not affect sol.

4. Equation: $y' - y = 0$. Suggested solution: $y = x^2 \Rightarrow y' = 2x, 2x - x^2 \stackrel{?}{=} 0$ Not for all $x \therefore$ no solution
5. Equation: $y' - y = 0$. Suggested solution: $y = e^x \Rightarrow y' = e^x, e^x - e^x \stackrel{?}{=} 0 \checkmark \therefore$ solution
6. Equation: $y'' + y = 0$. Suggested solution: $y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2, 2 + x^2 \stackrel{?}{=} 0$ No.
7. Equation: $y'' + y - 2 = 0$. Suggested solution: $y = x^2. 2 + x^2 - 2 \stackrel{?}{=} 0$ No
8. Equation: $y'' + 4y = 0$. Suggested solution: $y = \cos 4x \Rightarrow y' = -4 \sin 4x \Rightarrow y'' = -16 \cos 4x$
 $-16 \cos 4x + 4 \cos 4x \stackrel{?}{=} 0$ No
9. Equation: $y'' + y = 0$. Suggested solution: $y = \sin t + \cos t \Rightarrow y' = \cos t - \sin t \Rightarrow y'' = -\sin t - \cos t$
 $(-\sin t - \cos t) + (\sin t + \cos t) \stackrel{?}{=} 0$ Yes
10. Equation: $y'' + y = 0$. Suggested solution: $y = 3 \sin t + 2 \cos t \Rightarrow y' = 3 \cos t - 2 \sin t$
 $(-3 \sin t - 2 \cos t) + (3 \sin t + 2 \cos t) \stackrel{?}{=} 0$ Yes $y'' = -3 \sin t - 2 \cos t$
11. Equation: $y'' + 2y' + y = 0$. Suggested solution: $y = e^{-t} \Rightarrow y' = -e^{-t} \Rightarrow y'' = e^{-t}$
 $e^{-t} + 2(-e^{-t}) + e^{-t} \stackrel{?}{=} 0$ Yes
12. Equation: $y'' + 2y' + 2y = 0$. Suggested solution: $y = e^{-t} \sin t$.

¹ It may not be the only solution.

$$\begin{aligned} y' &= e^{-t} \cos t + (\sin t)(-e^{-t}) \\ y'' &= e^{-t}(-\sin t) + (\cos t)(-e^{-t}) \\ &\quad + (\sin t)(e^{-t}) + (-e^{-t})(\cos t) \\ &= -2e^{-t} \cos t \\ -2e^{-t} \cos t + (2e^{-t} \cos t - 2e^{-t} \sin t) + 2e^{-t} \sin t &\stackrel{?}{=} 0 \text{ Yes!} \end{aligned}$$

13. (10 marks) Studying predator-prey oscillations in class we encountered the differential

equation $h''(t) + ab h(t) = 0$ and its solution $h(t) = -v_0 \sqrt{\frac{a}{b}} \sin(\sqrt{ab} t) + u_0 \cos(\sqrt{ab} t)$

(presented a little differently on the whiteboard). Show that this is a correct solution.

$$h'(t) = -v_0 \sqrt{\frac{a}{b}} \cos(\sqrt{ab} t) \cdot \sqrt{ab} + u_0 (-\sin(\sqrt{ab} t) \cdot \sqrt{ab})$$

$$\begin{aligned} h''(t) &= v_0 \sqrt{\frac{a}{b}} \sin(\sqrt{ab} t) \sqrt{ab} \sqrt{ab} - u_0 \cos(\sqrt{ab} t) \sqrt{ab} \sqrt{ab} \\ &= v_0 a \sqrt{ab} \sin(\sqrt{ab} t) - ab u_0 \cos(\sqrt{ab} t) \end{aligned}$$

$$v_0 a \sqrt{ab} \sin \sqrt{ab} t - ab u_0 \cos \sqrt{ab} t + ab (-v_0) \sqrt{\frac{a}{b}} \sin \sqrt{ab} t + ab u_0 \cos \sqrt{ab} t \stackrel{?}{=} 0$$

Yes! $\therefore$ Solution!